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Introduction to the Continuum Mechanics of Solid Bodies

University of Miskolc, Hungary
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Preface

The present book is written for those Hungarian and international MSc and doctoral students who have to take a one-semester course in Continuum Mechanics. When I began to write it I faced with the following problems:

(1) What level of preliminary knowledge can be expected: Hungarian students in mechanical engineering with a BSc degree have some familiarity with tensor algebra and analysis. However, this is, in general, not the case for foreign students because of the differences in curricula. We had to compromise and devoted a chapter of introductory character to the fundamental concepts of tensor algebra and analysis.

(2) What notational system should be used: The direct notation has the advantage of providing more clarity and insight concerning the phenomenon investigated. In contrast, indicial notation makes possible to establish the component equations much easier. We had to compromise again and in many cases we applied the two notation systems parallel to each other.

(3) What parts of Continuum Mechanics should be included in the book: I remark that the present version of this book covers that part of Continuum Mechanics which can be studied within the framework of a one-semester course (two hours a week).

The text is organized into eight chapters. The first is devoted to the mathematical preliminaries. It contains the elements of tensor algebra and analysis with an emphasis on the issue how to use indicial notation in Cartesian coordinate systems.

Geometry of the non-linear deformations is presented in Chapter 2 which contains definitions of the deformation gradients and the various strain tensors. Furthermore we clarify how the deformation gradient can be decomposed and the most important strain measures are also considered.

Chapter 3 is concerned with the time derivatives of the most important kinematic quantities by introducing the concept of material time derivatives.

The results of kinematic linearization are gathered in Chapter 4. Within the framework of the linear theory we investigate what are the necessary and sufficient conditions the strains should meet in order to be compatible both in simply connected regions and in multiply connected ones.

Chapter 5 introduces the concept of the Cauchy stress tensor. Then various stress tensors are defined by utilizing pure mathematical transformations. It has also been shown what the extreme properties of the normal and shearing stresses are.

The fundamental laws solid bodies should obey are given in Chapter 6. They include the principle of mass conservation, the balance laws – the general and complete solution of the equilibrium equations is presented here – as well as the first and second theorems of thermodynamics.

Energy principles are treated in Chapter 7. After presenting the principle of virtual power and the principle of complementary virtual power we proceed with the principles of virtual work and a solution algorithm for the nonlinear problems is also is considered.

Chapter 8 is devoted to the constitutive theory with an emphasis on the nonlinear constitutive equations of elasticity. As regards the linear theory the heat effects are also taken into account in the generalized Hook's law.

Appendix A contains some longer mathematical transformations and clarifies the concept of isotropy.

Appendix B is a collection of the solutions to those problems presented at the ends of the various chapters.

Acknowledgments. Since this book is a textbook it is important to acknowledge my debt to one of my teachers. Prof. Imre Kozák introduced me to the modern literature on applied mechanics including the mechanics of solid bodies. I could always turn to him for advice when difficulties arose in my research work. In more than one case the text of the present book reflects the approach taken by Prof. Imre Kozák (1930-2016) to certain questions of solid mechanics.

I owe my wife Babi a debt of gratitude for her encouragement. Special thanks are due to my daughter Agnes and son Adam for their continuous support.

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György Szeidl

CHAPTER 1

Mathematical preliminaries (A review)

1.1. Vectors and vector operations

1.1.1. Coordinate system. The space under consideration is always a three dimensional Euclidean space. A Cartesian frame (Cartesian coordinate system) is determined by (a) an orthonormal basis $\mathbf{i}_x = \mathbf{i}_1$, $\mathbf{i}_y = \mathbf{i}_2$, and $\mathbf{i}_z = \mathbf{i}_3$, (b) a point O called origin, (c) and the three coordinate axes – see Figure 1.1 for details. Vectors are designated by boldface letters.

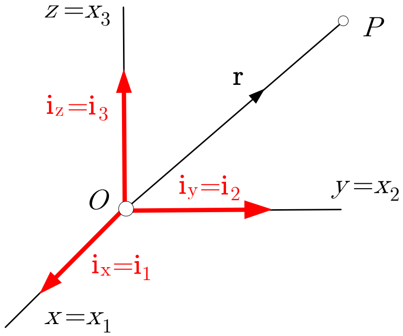

FIGURE 1.1. Coordinate systems

The letters $\mathbf{r}$ (or $\mathbf{x}$) denote the position vector of a point P with respect to the origin in the 3D space.

REMARK 1.1: Note that Figure 1.1 shows two coinciding Cartesian coordinate systems, namely (a) the coordinate system (xyz), and (b) the coordinate system $(x_1x_2x_3)$ for which it holds that

$$x = x_1\,, \quad y = x_2\,, \quad z = x_3 \qquad \text{and} \qquad \mathbf{i}_x = \mathbf{i}_1\,, \quad \mathbf{i}_y = \mathbf{i}_2\,, \quad \mathbf{i}_z = \mathbf{i}_3\,.$$

The vectors $\mathbf{i}_x = \mathbf{i}_1$, $\mathbf{i}_y = \mathbf{i}_2$ and $\mathbf{i}_z = \mathbf{i}_3$ are referred to as base vectors since any vector can be given in terms of the vectors $\mathbf{i}_x = \mathbf{i}_1$, $\mathbf{i}_y = \mathbf{i}_2$ and $\mathbf{i}_z = \mathbf{i}_3$.

1.1.2. Additive vector operations. A non zero vector $\mathbf{u}$ can be given in the coordinate system (xyz) [or in the coordinate system $(x_1x_2x_3)$] as

$$\mathbf{u} = u_x\mathbf{i}_x + u_y\mathbf{i}_y + u_z\mathbf{i}_z = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3 = \sum_{\ell=1}^{3} u_\ell\mathbf{i}_\ell = |\mathbf{u}|\,\mathbf{e}_u\,; \quad |\mathbf{e}_u| = 1\,, \tag{1.1a}$$

where

$$|\mathbf{u}| = \sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{\sum_{\ell=1}^{3} u_\ell^2} \tag{1.1b}$$

is the magnitude (the length of the vector $\mathbf{u}$) in a given unit (in mm for instance if $\mathbf{u}$ is a displacement vector), $\mathbf{e}_u$ is the direction vector of $\mathbf{u}$ while u_x, u_y, u_z and u_1, u_2, u_3 are the (scalar) components of $\mathbf{u}$ – see Figure 1.2. The point P where the vector $\mathbf{u}$ exerts its effect (where it works) is referred to as point of application. The straight line that contains $\mathbf{u}$ is the line of action of $\mathbf{u}$.

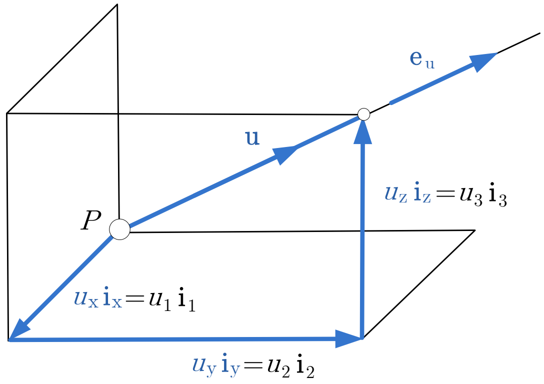

FIGURE 1.2. Components of the vector $\mathbf{u}$

The magnitude of a zero vector is zero. It has no direction (hence it may be regarded as if it were parallel or orthogonal to any non zero vector). The zero vectors are denoted by a boldface zero: $\mathbf{0}$.

For the position vector it holds that

$$\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = \sum_{\ell=1}^{3} x_\ell \mathbf{i}_\ell = \mathbf{x}\,. \tag{1.2}$$

A vector is called a [free]{fixed} vector if it [can be moved freely in space by preserving its length and orientation]{has a fixed point of application}.

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors and λ a scalar. Then

$$\boxed{\mathbf{u} \pm \mathbf{v} = (u_x \pm v_x)\,\mathbf{i}_x + (u_y \pm v_y)\,\mathbf{i}_y + (u_z \pm v_z)\,\mathbf{i}_z = \sum_{\ell=1}^{3} (u_\ell \pm v_\ell)\,\mathbf{i}_\ell} \tag{1.3a}$$

and

$$\boxed{\lambda\mathbf{u} = \lambda u_x \mathbf{i}_x + \lambda u_y \mathbf{i}_y + \lambda u_z \mathbf{i}_z = \sum_{\ell=1}^{3} \lambda u_\ell \mathbf{i}_\ell\,.} \tag{1.3b}$$

1.1.3. Dot product. Let $\varphi \in [0, \pi]$ be the angle formed by $\mathbf{u}$ and $\mathbf{v}$. The dot product of two vectors is defined as

$$\boxed{\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\varphi\,.} \tag{1.4}$$

The dot product has the following properties (λ and μ are scalars, $\mathbf{w}$ is a further vector)

$$\left.\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathbf{v}\cdot\mathbf{u}\,, \\ (\lambda\mathbf{u})\cdot(\mu\mathbf{v}) &= \lambda\mu\mathbf{v}\cdot\mathbf{u}\,, \\ \mathbf{w}\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{w}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{v}\,. \end{aligned}\right\} \tag{1.5}$$

Assume that $|\mathbf{u}|\neq 0$ and $|\mathbf{v}|\neq 0$. It follows from definition (1.4) that

$$\mathbf{u}\cdot\mathbf{v}=0 \quad \text{if} \;\; \varphi=\pi/2 \;\; (\mathbf{u}\cdot\mathbf{v}=0 \text{ is a condition of perpendicularity }) \tag{1.6a}$$

and

$$|\mathbf{u}|=\sqrt{\mathbf{u}\cdot\mathbf{u}}\,. \tag{1.6b}$$

Since the base vectors $\mathbf{i}_x=\mathbf{i}_1$, $\mathbf{i}_y=\mathbf{i}_2$ and $\mathbf{i}_z=\mathbf{i}_3$ constitute an orthonormal triplet it holds that

$$\mathbf{i}_n\cdot\mathbf{i}_m=\left\{\begin{array}{ll} 1 & m=n \\ 0 & m\neq n \end{array}\right| \qquad m,n=x,y,z\,, \quad (\text{or } \; m,n=1,2,3) \tag{1.7}$$

Consequently, for the dot product $\mathbf{u}\cdot\mathbf{v}$ we get

$$\mathbf{u}\cdot\mathbf{v}=(u_x\mathbf{i}_x+u_y\mathbf{i}_y+u_z\mathbf{i}_z)\cdot(v_x\mathbf{i}_x+v_y\mathbf{i}_y+v_z\mathbf{i}_z)=u_xv_x+u_yv_y+u_zv_z=\sum_{\ell=1}^{3}u_\ell v_\ell\,. \tag{1.8}$$

Making use of the dot product we can give the (scalar) components of the vectors $\mathbf{r}$ and $\mathbf{u}$ as

$$\begin{array}{llll} x=\mathbf{r}\cdot\mathbf{i}_x\,, & y=\mathbf{r}\cdot\mathbf{i}_y\,, & z=\mathbf{r}\cdot\mathbf{i}_z\,; & x_\ell=\mathbf{r}\cdot\mathbf{i}_\ell\,, \\ u_x=\mathbf{u}\cdot\mathbf{i}_x\,, & u_y=\mathbf{u}\cdot\mathbf{i}_y\,, & u_z=\mathbf{u}\cdot\mathbf{i}_z\,; & u_\ell=\mathbf{u}\cdot\mathbf{i}_\ell\,. \end{array} \tag{1.9}$$

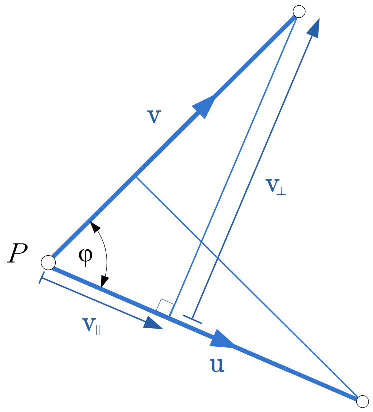

FIGURE 1.3. Resolution of a vector into a two perpendicular components

It follows from Figure 1.3 that a vector, say the vector $\mathbf{v}$, can be resolved into two components: one parallel to a given direction (say to the direction of the vector $\mathbf{u}$) and the other perpendicular to the given direction:

$$\mathbf{v}=\mathbf{v}_{||}+\mathbf{v}_{\perp}\,, \tag{1.10a}$$

where

$$\mathbf{v}_{||} = \mathbf{e}_u \, (\mathbf{v} \cdot \mathbf{e}_u) = \mathbf{e}_u \, |\mathbf{v}| \cos \varphi \,, \qquad \mathbf{e}_u = \frac{\mathbf{u}}{|\mathbf{u}|} \tag{1.10b}$$

is the component of **v** parallel to **u** and

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{||} = \mathbf{v} - \mathbf{e}_u \, (\mathbf{v} \cdot \mathbf{e}_u) \tag{1.10c}$$

is the component of **v** perpendicular to **u**.

1.1.4. Cross product. The cross product **w** of the two vectors **u** and **v** is written as

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} \tag{1.11a}$$

and is defined by the following properties:

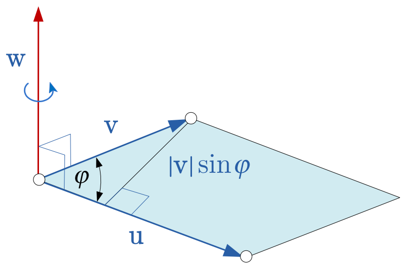

FIGURE 1.4. Cross product

(i) Assume that **u**, **v** and **w** have a common point of application. The line of action of **w** is perpendicular to the plane that contains **u** and **v** – see Figure 1.4.

(ii) The magnitude of **w** is given by

$$|\mathbf{w}| = |\mathbf{u}| \, |\mathbf{v}| \sin \varphi \tag{1.11b}$$

which is the area of the parallelogram in Figure 1.4.

(iii) The direction of **w** is such that an observer located at the tip of **w** will find as counterclockwise the rotation φ – see Figure 1.4 – which brings the first factor **u** into the second factor **v**. In other words the direction of **w** is given by the right-hand rule. If the vectors **u**, **v** and **w** do not have a common point of application they should first be redrawn from a common point.

Properties (**t** is a vector):

$$\left.\begin{aligned} (\lambda \mathbf{u}) \times (\mu \mathbf{v}) &= \lambda \mu \, \mathbf{u} \times \mathbf{v} \,, \\ \mathbf{t} \times (\mathbf{u} + \mathbf{v}) &= \mathbf{t} \times \mathbf{u} + \mathbf{t} \times \mathbf{v} \,, \\ \mathbf{u} \times \mathbf{v} &= -\mathbf{v} \times \mathbf{u} \,, \\ \mathbf{t} \times (\mathbf{u} \times \mathbf{v}) &= (\mathbf{v} \times \mathbf{u}) \times \mathbf{t} \,. \end{aligned}\right\} \tag{1.12}$$

It follows from the definition ($\mathbf{u} \neq \mathbf{0}$) that

$$\mathbf{u} \times \mathbf{u} = \mathbf{0} \,. \tag{1.13a}$$

Let $\mathbf{u}$ and $\mathbf{v}$ be non zero vectors. If

$$\mathbf{u} \times \mathbf{v} = \mathbf{0} \tag{1.13b}$$

then they are parallel to each other. Equation (1.13b) is known as the condition of parallelism.

On the basis of Figure 1.1 the definition of the cross product leads to the following relations

$$\begin{aligned} &\mathbf{i}_x \times \mathbf{i}_y = \mathbf{i}_z\ , \quad \mathbf{i}_y \times \mathbf{i}_z = \mathbf{i}_x\ , \quad \mathbf{i}_z \times \mathbf{i}_x = \mathbf{i}_y\ ; \\ &\mathbf{i}_1 \times \mathbf{i}_2 = \mathbf{i}_3\ , \quad \mathbf{i}_2 \times \mathbf{i}_3 = \mathbf{i}_1\ , \quad \mathbf{i}_3 \times \mathbf{i}_1 = \mathbf{i}_2\ . \end{aligned} \tag{1.14}$$

If these relations are satisfied the vectors $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ form a right hand triad.

Making use of relations (1.14) we can check with ease that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_x\mathbf{i}_x + u_y\mathbf{i}_y + u_z\mathbf{i}_z) \times (v_x\mathbf{i}_x + v_y\mathbf{i}_y + v_z\mathbf{i}_z) = \\ &= (u_y v_z - u_z v_y)\,\mathbf{i}_x + (u_z v_x - u_x v_z)\,\mathbf{i}_y + (u_x v_y - u_y v_x)\,\mathbf{i}_z \end{aligned} \tag{1.15a}$$

or

$$\boxed{\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} .} \tag{1.15b}$$

1.1.5. Triple cross product. It can be shown that the triple cross products $\mathbf{t}\times(\mathbf{u} \times \mathbf{v})$ and $(\mathbf{t} \times \mathbf{u}) \times \mathbf{v}$ can be calculated as

$$\boxed{\begin{aligned} \mathbf{t}\times(\mathbf{u} \times \mathbf{v}) &= (\mathbf{t}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{t}\cdot\mathbf{u})\,\mathbf{v}\,, \\ (\mathbf{t} \times \mathbf{u}) \times \mathbf{v} &= (\mathbf{t}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{t}\,. \end{aligned}} \tag{1.16}$$

Making use of the triple cross product we can also give the component of the vector $\mathbf{v}$ perpendicular to the direction $\mathbf{e}_u$ in the form

$$\mathbf{v}_\perp = \mathbf{e}_u \times (\mathbf{v} \times \mathbf{e}_u) = (\mathbf{e}_u \times \mathbf{v}) \times \mathbf{e}_u = \mathbf{v} - \mathbf{e}_u\,(\mathbf{v}\cdot\mathbf{e}_u)\ . \tag{1.17}$$

1.1.6. Box product. The box product of the vectors $\mathbf{t}$, $\mathbf{u}$ and $\mathbf{v}$ is defined by the equation

$$\boxed{[\mathbf{t}\,\mathbf{u}\,\mathbf{v}] = \mathbf{t}\cdot(\mathbf{u} \times \mathbf{v})\ .} \tag{1.18}$$

Properties of the box product:

(i) By applying rule (1.15) for the cross product $\mathbf{u} \times \mathbf{v}$ and then rule (1.8) for the dot product $\mathbf{t}\cdot(\mathbf{u} \times \mathbf{v})$ we get

$$\mathbf{t}\cdot(\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} t_x & t_y & t_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} v_x & v_y & v_z \\ t_x & t_y & t_z \\ u_x & u_y & u_z \end{vmatrix} = (\mathbf{t} \times \mathbf{u})\cdot\mathbf{v}\ . \tag{1.19a}$$

where we have taken into account that a determinant remains unchanged if its rows are appropriately interchanged. Equation (1.19a) shows that the operation symbols dot and cross are interchangeable.

(ii) If we now utilize (a) the previous equation and (b) the commutativity of the scalar product we get

$$[\mathbf{t}\,\mathbf{u}\,\mathbf{v}] = [\mathbf{u}\,\mathbf{v}\,\mathbf{t}] = [\mathbf{v}\,\mathbf{t}\,\mathbf{u}]\ . \tag{1.19b}$$

This rule is the cyclic interchangeability of the factors in the box product. If we take equation $(1.12)_3$ also into account we obtain

$$[\mathbf{t}\,\mathbf{u}\,\mathbf{v}] = -[\mathbf{u}\,\mathbf{t}\,\mathbf{v}] = -[\mathbf{v}\,\mathbf{u}\,\mathbf{t}] = -[\mathbf{t}\,\mathbf{v}\,\mathbf{u}]\ . \tag{1.19c}$$

(iii) Figure 1.5 shows that the absolute value of the box product is the volume of the parallelepiped determined by the vectors $\mathbf{t}$, $\mathbf{u}$ and $\mathbf{v}$ if they are drawn from a common point.

(iv) If $[\mathbf{t}\,\mathbf{u}\,\mathbf{v}] = 0$ then the vectors $\mathbf{t}$, $\mathbf{u}$ and $\mathbf{v}$ drawn from the same point lie in a plane.

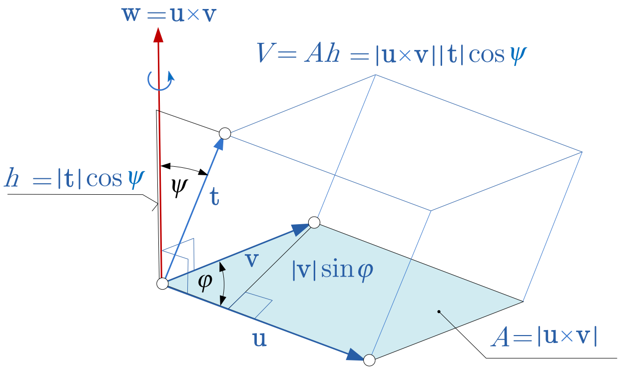

FIGURE 1.5. Geometry of the box product

1.1.7. Linearly independent vectors. Let v^1, v^2 and v^3 be three scalars. The vectors $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ ($|\mathbf{g}_\ell| \neq 0\ \ell = 1,2,3$) are said to be linearly independent if the vectorial equation

$$v^1\mathbf{g}_1 + v^2\mathbf{g}_2 + v^3\mathbf{g}_3 = \mathbf{0} \tag{1.20a}$$

has only the trivial solution $v^1 = v^2 = v^3 = 0$ for the scalars v_ℓ. For the linearly independent vector triplet $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ the determinant of the corresponding scalar equations

$$\begin{bmatrix} g_{x1} & g_{x2} & g_{x3} \\ g_{y1} & g_{y2} & g_{y3} \\ g_{z1} & g_{z2} & g_{z3} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.20b}$$

is different from zero. Therefore, the vectorial equation

$$v^1\mathbf{g}_1 + v^2\mathbf{g}_2 + v^3\mathbf{g}_3 = \mathbf{v}\,, \tag{1.20c}$$

in which $\mathbf{v} = v_x\mathbf{i}_x + v_y\mathbf{i}_y + v_z\mathbf{i}_z \neq \mathbf{0}$ is a known vector, always has a unique solution for the scalar triplet v^ℓ ($\ell = 1,2,3$). These scalars are called the components of the vector $\mathbf{v}$ in the directions $\mathbf{g}_\ell$. Hence, the linearly independent vectors $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ form a basis of the 3D space: any vector $\mathbf{v}$ can be given in terms of $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$. For this reason they are also called base vectors.

Let us define the dual base vectors by the following equations

$$\boxed{\begin{gathered}\mathbf{g}^1 = \mathbf{g}_1^* = \frac{\mathbf{g}_2 \times \mathbf{g}_3}{\gamma_\circ}\,, \quad \mathbf{g}^2 = \mathbf{g}_2^* = \frac{\mathbf{g}_3 \times \mathbf{g}_1}{\gamma_\circ}\,, \quad \mathbf{g}^3 = \mathbf{g}_3^* = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\gamma_\circ}\,, \\ \gamma_\circ = [\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]\,,\end{gathered}} \tag{1.21}$$

where $\gamma_\circ$ is the determinant of equations (1.20). It is clear that $\mathbf{g}_k \perp \mathbf{g}^\ell$ if $\ell \neq k$. Consequently,

$$\boxed{\mathbf{g}_k \cdot \mathbf{g}^\ell = \mathbf{g}_k \cdot \mathbf{g}_\ell^* = \begin{cases} 1 & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell, \end{cases} \qquad k, \ell = 1, 2, 3} \tag{1.22}$$

where we have taken into account that $\mathbf{g}_k \cdot \mathbf{g}^k = \gamma_\circ$ $(k = 1, 2, 3)$.

It is not too difficult to check by using the expansion rule (1.16) valid for the triple cross product that

$$\mathbf{g}^2 \times \mathbf{g}^3 = (\mathbf{g}_3 \times \mathbf{g}_1) \times (\mathbf{g}_1 \times \mathbf{g}_2)/\gamma_o^2 = \{\underbrace{[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]}_{=\gamma_o}\mathbf{g}_1 - \underbrace{[\mathbf{g}_1\mathbf{g}_1\mathbf{g}_2]}_{=0}\mathbf{g}_3\}/\gamma_o^2\,. \tag{1.23a}$$

Hence

$$\mathbf{g}_1 = \left(\mathbf{g}^2 \times \mathbf{g}^3\right)\gamma_o\,, \quad 1 = \mathbf{g}^1 \cdot \mathbf{g}_1 = \left[\mathbf{g}^1\mathbf{g}^2\mathbf{g}^3\right]\gamma_o\,,$$

that is,

$$\mathbf{g}_1 = \frac{\mathbf{g}^2 \times \mathbf{g}^3}{\frac{1}{\gamma_o}} = \frac{\mathbf{g}^2 \times \mathbf{g}^3}{\left[\mathbf{g}^1\mathbf{g}^2\mathbf{g}^3\right]}\,. \tag{1.23b}$$

This result shows the dual vector of $\mathbf{g}^1$ is the original vector $\mathbf{g}_1$. It can be proved in the same way that this statement is valid for the other two cases as well.

Since $\left[\mathbf{g}^1\mathbf{g}^2\mathbf{g}^3\right] \neq 0$ it follows that the dual base vectors form also a basis in the 3D space.

Consider now the dot product $\mathbf{v} \cdot \mathbf{g}^k$. With regard to equation (1.22) we get

$$\mathbf{v} \cdot \mathbf{g}^k = \left(v^1\mathbf{g}_1 + v^2\mathbf{g}_2 + v^3\mathbf{g}_3\right) \cdot \mathbf{g}^k = v^1\mathbf{g}_1 \cdot \mathbf{g}^k + v^2\mathbf{g}_2 \cdot \mathbf{g}^k + v^3\mathbf{g}_3 \cdot \mathbf{g}^k = v^k$$

thus

$$\boxed{\mathbf{v} \cdot \mathbf{g}^k = v^k\,.} \tag{1.24}$$

This equation is a pair (or generalization) of equation (1.9) valid in Cartesian coordinate systems. It says that the component of the vector $\mathbf{v}$ in the direction $\mathbf{g}_k$ can be obtained if we dot multiply the vector $\mathbf{v}$ by the dual pair of the vector $\mathbf{g}_k$, i.e., by $\mathbf{g}^k = \mathbf{g}_k^*$.

In Cartesian coordinate systems the base vectors and the dual base vectors are the same: $\mathbf{i}_m = \mathbf{i}_m^*$ $(m = x, y, z$ or $1, 2, 3)$.

1.2. Transformation rules

Assume that we have two Cartesian coordinate systems namely the coordinate systems $(x_1\, x_2\, x_3)$ and $(x'_1\, x'_2\, x'_3)$ which have the same origin – see Figure 1.6. The base vectors are denoted by $\mathbf{i}_k$ and $\mathbf{i}'_\ell$ $(k, \ell = 1, 2, 3)$, respectively.

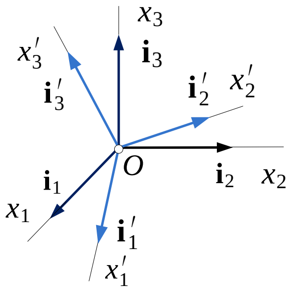

FIGURE 1.6. Two Cartesian coordinate systems

A vector $\mathbf{u}$ can be given in both coordinate systems:

$$\mathbf{u} = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3 = u'_1\mathbf{i}'_1 + u'_2\mathbf{i}'_2 + u'_3\mathbf{i}'_3\,. \tag{1.25}$$

Assume that $\mathbf{u}$ is known in the unprimed coordinate system. Assume further that the unit vectors $\mathbf{i}'_\ell$ are also known in terms of the unit vectors $\mathbf{i}_k$. Recalling $(1.9)_2$ the scalar component u'_ℓ can be calculated as

$$u'_\ell = \mathbf{u}\cdot\mathbf{i}'_\ell = \sum_{k=1}^{3} \mathbf{i}'_\ell \cdot (\mathbf{i}_k u_k) = \sum_{k=1}^{3} \underbrace{(\mathbf{i}'_\ell \cdot \mathbf{i}_k)}_{Q_{\ell' k}} u_k = \sum_{k=1}^{3} Q_{\ell' k}\, u_k\,, \qquad \ell = \ell' = 1, 2, 3\,. \tag{1.26a}$$

The dot product

$$\boxed{\mathbf{i}'_\ell \cdot \mathbf{i}_k = Q_{\ell' k} \qquad \ell, k = 1, 2, 3} \tag{1.26b}$$

is the scalar component of the vector $\mathbf{i}'_\ell$ in the coordinate direction k. Consequently, it holds that

$$\mathbf{i}'_\ell = \sum_{k=1}^{3} Q_{\ell' k}\, \mathbf{i}_k\,. \qquad \ell' = 1, 2, 3\,. \tag{1.26c}$$

By introducing the matrices

$$[Q_{\ell' k}] = \underset{(3\times 3)}{\underline{\mathbf{Q}}} = \begin{bmatrix} Q_{1'1} & Q_{1'2} & Q_{1'3} \\ Q_{2'1} & Q_{2'2} & Q_{2'3} \\ Q_{3'1} & Q_{3'2} & Q_{3'3} \end{bmatrix} = \begin{bmatrix} \mathbf{i}'_1\cdot\mathbf{i}_1 & \mathbf{i}'_1\cdot\mathbf{i}_2 & \mathbf{i}'_1\cdot\mathbf{i}_3 \\ \mathbf{i}'_2\cdot\mathbf{i}_1 & \mathbf{i}'_2\cdot\mathbf{i}_2 & \mathbf{i}'_2\cdot\mathbf{i}_3 \\ \mathbf{i}'_3\cdot\mathbf{i}_1 & \mathbf{i}'_3\cdot\mathbf{i}_2 & \mathbf{i}'_3\cdot\mathbf{i}_3 \end{bmatrix},$$

$$\underset{(3\times 1)}{\underline{\mathbf{u}}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad \underset{(3\times 1)}{\underline{\mathbf{u}}'} = \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} \tag{1.27}$$

equation (1.26a) can be rewritten in a matrix form:

$$\boxed{\underset{(3\times1)}{\underline{\mathbf{u}}'} = \underset{(3\times3)}{\underline{\mathbf{Q}}}\ \underset{(3\times1)}{\underline{\mathbf{u}}}\ ,} \tag{1.28}$$

in which $\underline{\mathbf{Q}}$ is the transformation matrix. If $\underline{\mathbf{u}}$ and $\underline{\mathbf{Q}}$ are known equation (1.28) makes possible to determine the scalar components u'_ℓ (or $\underline{\mathbf{u}}'$) in the primed coordinate system.

If the scalar components u'_ℓ are known and the scalar components u_k are the unknowns the procedure is similar:

$$u_k = \mathbf{u}\cdot\mathbf{i}_k = \sum_{\ell=1}^{3}\mathbf{i}_k\cdot(\mathbf{i}'_\ell u'_\ell) = \sum_{\ell=1}^{3}\underbrace{(\mathbf{i}_k\cdot\mathbf{i}'_\ell)}_{Q_{k\ell'}}u'_\ell = \sum_{\ell=1}^{3}Q_{k\ell'}\,u'_\ell\,,\qquad k=1,2,3\,. \tag{1.29a}$$

Here

$$\boxed{\mathbf{i}_k\cdot\mathbf{i}'_\ell = Q_{k\ell'}\,,\qquad \ell,k=1,2,3\,.} \tag{1.29b}$$

It also holds that

$$\mathbf{i}_k = \sum_{\ell=1}^{3}Q_{k\ell'}\,\mathbf{i}'_\ell \qquad k=1,2,3\,. \tag{1.29c}$$

The reasoning is similar to that given for equation (1.26c).

Making use of the notations introduced by equation (1.27) transformation (1.29a) can also be given in a matrix form

$$\boxed{\underset{(3\times1)}{\underline{\mathbf{u}}} = \underset{(3\times3)}{\underline{\mathbf{Q}}^T}\ \underset{(3\times1)}{\underline{\mathbf{u}}'}\ .} \tag{1.30}$$

where ^{T} is the symbol for transpose.

Upon substitution of equation (1.30) into (1.28) we get

$$\underset{(3\times1)}{\underline{\mathbf{u}}'} = \underset{(3\times3)}{\underline{\mathbf{Q}}}\ \underset{(3\times1)}{\underline{\mathbf{u}}} = \underset{(3\times3)}{\underline{\mathbf{Q}}}\ \underbrace{\underset{(3\times3)}{\underline{\mathbf{Q}}^T}\ \underset{(3\times1)}{\underline{\mathbf{u}}'}}_{\underset{(3\times1)}{\underline{\mathbf{u}}}}\ .$$

Here the right side can be equal to $\underline{\mathbf{u}}'$ if and only if

$$\boxed{\underset{(3\times3)}{\underline{\mathbf{Q}}}\ \underset{(3\times3)}{\underline{\mathbf{Q}}^T} = \underset{(3\times3)}{\underline{\mathbf{1}}}\ ,\qquad \underset{(3\times3)}{\underline{\mathbf{1}}} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}} \tag{1.31}$$

from where it follows that $\underline{\mathbf{Q}}^T$ is the inverse of $\underline{\mathbf{Q}}$:

$$\underline{\mathbf{Q}}^T = \underline{\mathbf{Q}}^{-1}\,. \tag{1.32}$$

If we take the determinant of the product $\underline{\mathbf{Q}}\,\underline{\mathbf{Q}}^T$ we obtain from equation (1.31) that

$$\det\left(\underline{\mathbf{Q}}\,\underline{\mathbf{Q}}^T\right) = \det\left(\underline{\mathbf{Q}}\right)\,\det\left(\underline{\mathbf{Q}}^T\right) = \left[\det\left(\underline{\mathbf{Q}}\right)\right]^2 = 1\,.$$

Thus

$$\boxed{\det\left(\underline{\mathbf{Q}}\right) = \pm 1\,.} \tag{1.33}$$

A matrix whose transpose is equal to its inverse is referred to as orthogonal matrix. If, in addition, the determinant of this matrix is equal to 1 then the name is proper orthogonal.

REMARK 1.2: The matrix $\underline{\mathbf{Q}}$ is proper orthogonal. This statements is a consequence of the results presented later in Subsection 1.4.5 – proper orthogonal matrices belong to rotations.

EXERCISE 1.1: Figure 1.7 shows the coordinate systems $(x_1\,x_2\,x_3)$ and $(x_1'\,x_2'\,x_3')$. Determine the matrix $\underline{\mathbf{Q}}$.

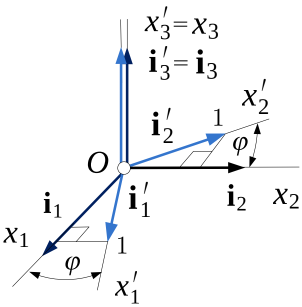

FIGURE 1.7. Rotation about the coordinate axis x_3

It is clear from Figure 1.7 that

$$\mathbf{i}_1' = \cos\varphi\,\mathbf{i}_1 + \sin\varphi\,\mathbf{i}_2\,, \qquad \mathbf{i}_2' = -\sin\varphi\,\mathbf{i}_1 + \cos\varphi\,\mathbf{i}_2\,, \qquad \mathbf{i}_3' = \mathbf{i}_3\,.$$

Given the base vectors equation $(1.27)_1$ yields

$$\underset{(3\times3)}{\underline{\mathbf{Q}}} = \begin{bmatrix} \mathbf{i}_1'\cdot\mathbf{i}_1 & \mathbf{i}_1'\cdot\mathbf{i}_2 & \mathbf{i}_1'\cdot\mathbf{i}_3 \\ \mathbf{i}_2'\cdot\mathbf{i}_1 & \mathbf{i}_2'\cdot\mathbf{i}_2 & \mathbf{i}_2'\cdot\mathbf{i}_3 \\ \mathbf{i}_3'\cdot\mathbf{i}_1 & \mathbf{i}_3'\cdot\mathbf{i}_2 & \mathbf{i}_3'\cdot\mathbf{i}_3 \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\det\underline{\mathbf{Q}} = \sin^2\varphi + \cos^2\varphi = 1$ the above matrix is proper orthogonal.

EXERCISE 1.2: Show that the matrix

$$\underset{(3\times3)}{\underline{\mathbf{Q}}} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{6} & \sqrt{2/3} & -1/\sqrt{6} \end{bmatrix}$$

is proper orthogonal.

Since

$$\underset{(3\times3)}{\underline{\mathbf{Q}}}\;\underset{(3\times3)}{\underline{\mathbf{Q}}}^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{6} & \sqrt{2/3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & \sqrt{2/3} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{(3\times 3)}{\underline{\mathbf{1}}} \qquad (1.34)$$

and

$$\det\left(\underline{\mathbf{Q}}\right) = \begin{vmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{6} & \sqrt{2/3} & -1/\sqrt{6} \end{vmatrix} =$$

$$= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{6}}\right) + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} = \frac{1}{3} + \left(\frac{1}{6} + \frac{1}{6}\right) + \frac{1}{3} = 1$$

it follows that $\underline{\mathbf{Q}}$ is proper orthogonal.

1.3. Indicial notation

1.3.1. Subscripts. In the sequel we shall assume that the [Latin]{Greek} character indices (subscripts) have the range [1,2,3]{1,2} for each index (subscript). If each quantity in an equation has the same indices (subscripts) then that equation holds over the range of those values the indices (subscripts) have:

$$u_r + v_r = w_r \qquad \Longleftrightarrow \qquad \begin{aligned} u_1 + v_1 &= w_1\,, \\ u_2 + v_2 &= w_2\,, \\ u_3 + v_3 &= w_3\,. \end{aligned} \quad \text{(3 equations)}$$

$$u_\alpha + v_\alpha = w_\alpha \qquad \Longleftrightarrow \qquad \begin{aligned} u_1 + v_1 &= w_1\,, \\ u_2 + v_2 &= w_2\,. \end{aligned} \quad \text{(2 equations)}$$

$$a_{k\ell} = b_k c_\ell \qquad \Longleftrightarrow \qquad \begin{aligned} a_{11} &= b_1 c_1\,, \\ a_{12} &= b_1 c_2\,, \\ &\dots \\ a_{33} &= b_3 c_3\,. \end{aligned} \quad \text{(9 equations)}$$

1.3.2. Summation convention. If an index appears twice in a term of an equation written using indicial notation summation over the range of the index is implied. Such an index is called dummy index (dummy index pair):

$$a_{k\ell}\, x_\ell = f_k \qquad \Longleftrightarrow \qquad \begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 &= f_1\,, \\ a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 &= f_2\,, \\ a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 &= f_3\,. \end{aligned} \qquad (1.35)$$

The index k here is a free index (a free index appears only once in each quantity in the equation considered) while ℓ is a dummy index. The general rules for an equation written in indicial notations are as follows [84]:

(i) The same free index (indices if the number of free indices is more than one) must appear in every term of the equation.

(ii) The relative position of the free indices is important:

$$a_{k\ell} = b_k c_\ell \qquad \text{is not the same as} \qquad a_{k\ell} = b_\ell c_k$$

since, for instance, $a_{12} = b_1 c_2$ and $a_{12} = b_2 c_1$ are different. When writing an equation in indicial notation it is, therefore, worth preserving the order of free indices in every term in the equation. The characters used for denoting free indices do not matter since the equations

$$a_{k\ell} = b_k c_\ell \qquad \text{and} \qquad a_{pq} = b_p c_q$$

are the same.

(iii) The free and dummy indices must be different. Equation

$$a_{\ell\ell}\, x_\ell = f_\ell$$

is meaningless.

(iv) The dummy index pairs must also be different. The quantity $d_{\ell\ell kk}$ is correct, while d_{kkkk} is mistaken (meaningless).

It follows from the above rules that an index may not appear more than twice in every term of an equation.

Exercise 1.3: Rewrite equations (1.26a) and (1.29a) using indicial notation.

The solution is obtained by canceling the summation symbols form the equations cited:

$$u'_\ell = Q_{\ell' k}\, u_k\,, \qquad u_k = Q_{k\ell'}\, u'_\ell\,. \tag{1.36}$$

1.3.3. Kronecker delta. The Kronecker delta operator[1] is defined by the equation

$$\boxed{\delta_{k\ell} = \delta_{\ell k} = \begin{cases} 1 & \text{if } k = \ell\,, \\ 0 & \text{if } k \neq \ell\,. \end{cases}} \tag{1.37a}$$

Its values can be displayed by the matrix equation

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\mathbf{1}}\,. \tag{1.37b}$$

Expanding the expression $\delta_{k\ell} a_\ell$ we get

$$\delta_{k\ell} a_\ell = \delta_{k1} a_1 + \delta_{k2} a_2 + \delta_{k3} a_3 = \left\{ \begin{array}{ll} a_1 & \text{if } k = 1 \\ a_2 & \text{if } k = 2 \\ a_3 & \text{if } k = 3 \end{array} \right\} = a_k\,.$$

Or briefly

$$\delta_{k\ell} a_\ell = a_k\,. \tag{1.38}$$

For this reason the Kronecker delta is also referred to as index renaming operator.

It follows from equation (1.7) that

$$\mathbf{i}_k \cdot \mathbf{i}_\ell = \delta_{k\ell}\,, \qquad \delta_{kk} = 3\,. \tag{1.39}$$

[1] Leopold Kronecker (1823-1891)

Making use of the above result and taking into account that $\delta_{k\ell}$ is an index renaming operator we can determine the dot product of the vectors $\mathbf{u}$ and $\mathbf{v}$ in the coordinate system $(x_1\, x_2\, x_3)$:

$$\mathbf{u}\cdot\mathbf{v} = \underbrace{u_k\,\mathbf{i}_k}_{\mathbf{u}}\cdot\underbrace{v_\ell\,\mathbf{i}_\ell}_{\mathbf{v}} = u_k\,v_\ell\,\underbrace{(\mathbf{i}_k\cdot\mathbf{i}_\ell)}_{\delta_{k\ell}} = u_k\,v_\ell\,\delta_{k\ell} = u_k\,v_k = u_\ell\,v_\ell\,. \tag{1.40}$$

1.3.4. Permutation symbol. The permutation symbol $e_{k\ell r}$ introduced by Tullio Levi-Civita[2] is defined by the following equation [89]:

$$\boxed{e_{k\ell r} = \begin{cases} 1 & \text{if } k\ell r = 123, 231, 312\,, \\ -1 & \text{if } k\ell r = 213, 321, 132\,, \\ 0 & \text{if two or more subscripts are equal.} \end{cases}} \tag{1.41}$$

Observe that

$$\begin{matrix} 123, 231, 312 \\ 213, 321, 132 \end{matrix} \quad \text{are} \quad \begin{matrix} \text{even} \\ \text{odd} \end{matrix} \quad \text{permutations of the numbers 1,2,3}\,.$$

Observe further that the permutations are the same for the triplets

$$\underbrace{k\ell r,\ \ell r k,\ r k \ell}_{\text{If these are [even]\{odd\}}} \quad \text{and} \quad \underbrace{\ell k r,\ r \ell k,\ k r \ell\,.}_{\text{these are [odd]\{even\}.}}$$

If these are [even]{odd} then these are [odd]{even}.

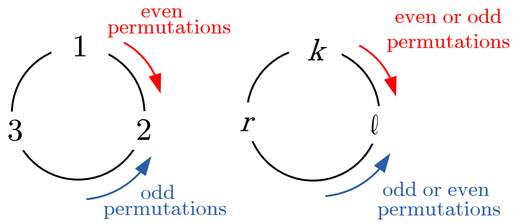

FIGURE 1.8. Circles for representing the permutations

Note that [even] (odd) permutations are obtained when the numbers $1, 2, 3$ are considered in [cyclic](reverse cyclic) order on the first circle shown in Figure 1.8. As regards the triplet $k\ell r$ the permutataions obtainend in [cyclic] order on the second circle are either (even) or [odd]; while the permutations in reverse cyclic order are (odd) or [even].

On the basis of this observation and the definition given by equation (1.41) we obtain

$$e_{k\ell r} = e_{\ell r k} = e_{r k \ell} = -e_{\ell k r} = -e_{r \ell k} = -e_{k r \ell}\,. \tag{1.42}$$

(If the order of two neighbouring subscripts is interchanged the permutation symbol changes its sign.)

It can be checked with ease by utilizing the properties of the permutation symbol that

$$\mathbf{i}_k \times \mathbf{i}_\ell = e_{k\ell r}\mathbf{i}_r\,. \tag{1.43}$$

[2]Tullio Levi-Civita (1873-1941)

This equation is equivalent to equations (1.14)$_2$ (to three equations). Consequently,

$$\mathbf{w} = w_r \mathbf{i}_r = \mathbf{u} \times \mathbf{v} = (u_k \mathbf{i}_k) \times (v_\ell \mathbf{i}_\ell) = u_k v_\ell \, \mathbf{i}_k \times \mathbf{i}_\ell = e_{k\ell r} u_k v_\ell \, \mathbf{i}_r = \underbrace{e_{rk\ell} u_k v_\ell}_{w_r} \mathbf{i}_r \tag{1.44a}$$

or simply

$$w_r = e_{rk\ell} u_k v_\ell \, . \tag{1.44b}$$

Remark 1.3: When writing vectorial equations in indicial notation it is customary to omit the base vectors in the same manner as we did it for equation (1.44b).

Using (1.18) the box product of the vectors $\mathbf{t}$, $\mathbf{u}$ and $\mathbf{v}$ in indicial notation can be written in the following form:

$$[\mathbf{t}\,\mathbf{u}\,\mathbf{v}] = \mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = t_s \mathbf{i}_s \cdot (u_k \mathbf{i}_k \times v_\ell \mathbf{i}_\ell) \underset{(1.44a)}{=\uparrow=} t_s \mathbf{i}_s \cdot (e_{rk\ell} u_k v_\ell \mathbf{i}_r) =$$
$$= t_s u_k v_\ell \, e_{rk\ell} \underbrace{\mathbf{i}_s \cdot \mathbf{i}_r}_{\delta_{sr}} = t_s u_k v_\ell \underbrace{\delta_{sr} e_{rk\ell}}_{\text{index renaming}} = e_{rk\ell} \, t_r u_k v_\ell \, . \tag{1.45}$$

1.3.5. Determinant. The determinant of $a_{k\ell}$ is given by the equation

$$|a_{k\ell}| = \det(a_{k\ell}) = e_{pqr} a_{1p} a_{2q} a_{3r} =$$
$$= a_{11} \left(e_{1qr} a_{2q} a_{3r} \right) + a_{12} \left(e_{2qr} a_{2q} a_{3r} \right) + a_{13} \left(e_{3qr} a_{2q} a_{3r} \right) =$$
$$= a_{11} \left(a_{22} a_{33} - a_{23} a_{32} \right) + a_{12} \left(a_{23} a_{13} - a_{21} a_{33} \right) + a_{13} \left(a_{21} a_{32} - a_{22} a_{31} \right) \, .$$

This expression is really the expansion of the determinant by the first row. If we multiply the equation

$$\boxed{|a_{k\ell}| = e_{pqr} a_{1p} a_{2q} a_{3r}} \tag{1.46}$$

by $e_{123} = 1$ from left and by 1 from right we get

$$e_{[1][2][3]} |a_{k\ell}| = e_{pqr} a_{[1]p} \, a_{[2]q} \, a_{[3]r} \, .$$

where square brackets are used to emphasize the role of the corresponding indices. Let us now substitute i for 1, j for 2 and k for 3 so that we can establish the more general relationship

$$\boxed{e_{ijk} |a_{k\ell}| = e_{pqr} a_{ip} a_{jq} a_{kr} = e_{ijk} e_{pqr} a_{1p} a_{2q} a_{3r} \, .} \tag{1.47}$$

By expanding the determinant and performing some paper and pencil calculations we can verify that

$$e_{pyk} e_{\ell dr} = \begin{vmatrix} \delta_{p\ell} & \delta_{pd} & \delta_{pr} \\ \delta_{y\ell} & \delta_{yd} & \delta_{yr} \\ \delta_{k\ell} & \delta_{kd} & \delta_{kr} \end{vmatrix} \, . \tag{1.48}$$

Exercise 1.4: Prove the following relations:

$$\boxed{\begin{aligned} e_{k\ell m} e_{kqr} &= \delta_{\ell q} \delta_{mr} - \delta_{\ell r} \delta_{mq} \, , \\ e_{k\ell m} e_{k\ell r} &= 2 \delta_{mr} \, , \\ e_{k\ell m} e_{k\ell m} &= 6 \, . \end{aligned}} \tag{1.49}$$

(i) Since the second and third relations follow from the first one it is worthy of beginning with the first. After expanding its left side we obtain

$$e_{k\ell m}e_{kqr} = e_{1\ell m}e_{1qr} + e_{2\ell m}e_{2qr} + e_{3\ell m}e_{3qr}\,.$$

This expression is different from zero if

(a) $\ell = q$, $m = r$ and $\ell \neq m$ (then that term is not zero on the right side for which the three subscripts differ from each other, and its value is 1);

(b) $\ell = r$, $m = q$ and $\ell \neq m$ (then that term is not zero on the right side for which the three subscripts differ from each other, its value is now -1)

since (a) the subscript triplets $_{k\,\ell=q\,m=r}$ and $_{kqr}$ have the same permutations (this can be odd or even) (b) the subscript triplets $_{k\,\ell=r\,m=q}$ and $_{kqr}$ have different permutations (if the first is even, the second is odd and conversely).

It can be checked with ease that for case (a) and (b) the right side

$$\delta_{\ell q}\delta_{mr} - \delta_{\ell r}\delta_{mq}$$

is equal to (a) 1, (b) -1; otherwise it is zero. The left and right sides have, therefore, the same value.

That was to be proved.

(ii) Since relation $(1.49)_1$ has been proved it is sufficient to examine what value the right side of $(1.49)_1$ has for $q = \ell$. We get

$$\underbrace{\delta_{\ell\ell}}_{=3}\delta_{mr} - \underbrace{\delta_{\ell r}\delta_{m\ell}}_{=\delta_{mr}} = 2\delta_{mr}\,.$$

(iii) Fulfillment of $(1.49)_3$ is now obvious.

A comparison of (1.47) and (1.49) yields:

$$|a_{k\ell}| = e_{ijk}e_{pqr}a_{ip}a_{jq}a_{kr}\,. \tag{1.50}$$

1.3.6. Contraction. If we make two (or more) free indices equal – see Exercise 1.4 – then we speak about contraction. Consider the product

$$a_{k\ell} = b_k c_\ell\,. \tag{1.51a}$$

For $k = \ell$ we get

$$a_{\ell\ell} = b_\ell c_\ell = b_1c_1 + b_2c_2 + b_3c_3\,. \tag{1.51b}$$

If b_ℓ and c_ℓ are the (scalar) components of the vectors $\mathbf{b}$ and $\mathbf{c}$ then this contraction is nothing but the scalar product $\mathbf{b}\cdot\mathbf{c}$. A further example for the application of contraction is presented here

$$\sigma_{pq} = C_{pqrs}\varepsilon_{rs} \quad \Leftrightarrow \quad \sigma_{pp} = C_{pprs}\varepsilon_{rs}\,, \tag{1.52}$$

where σ_{pq}, ε_{rs} and C_{pqrs} are two and four index quantities.

1.4. Tensor algebra

1.4.1. Scalars and vectors.

1.4.1.1. *Scalars.* A scalar quantity does not change its value if we rotate the coordinate system. For this property scalars are called tensors of zero order. It is worthy of mentioning that the independence of the coordinate system as regards the quantity considered is a fundamental requirement here and in the sequel as well.

1.4.1.2. *Vectors.* A vector (a vectorial quantity) is called first order tensor (tensor of order one). We use the expression first order tensor since we assume that each vector behaves like a radius (position) vector, i.e., it keeps its direction and magnitude when we rotate the coordinate system about the origin. Let the vector $\mathbf{t}$ be given in the unprimed and primed coordinate system:

$$\mathbf{t} = t_m \mathbf{i}_m = t'_\ell \, \mathbf{i}'_\ell \; . \tag{1.53}$$

The scalar triplets t_m and t'_ℓ are called (by definition) vectors if they follow the transformation rules

$$\boxed{t'_\ell = Q_{\ell' m} t_m \,, \qquad t_m = Q_{m\ell'} t'_\ell \,.} \tag{1.54}$$

REMARK 1.4: This definition is based on equations (1.26a) and (1.29a).

If these transformation rules are satisfied then the vector $\mathbf{t}$ is the same in both coordinate systems, i.e., it does not change its direction and magnitude when we rotate the coordinate axes about the origin.

1.4.2. Tensors of order two.

1.4.2.1. *Vector-vector functions.* A vector-vector function $\boldsymbol{\phi}$ assigns a set of vectors (the set is denoted by $\mathcal{B}$, the vectors that constitute the set by $\mathbf{w}$) to each member of another set of vectors (this set is denoted by $\mathcal{A}$, the vectors that constitute the set by $\mathbf{v}$). Assume that $\mathcal{A}$ is the set of the position vectors in the 3D space and the vectors $\mathbf{v}$ that constitute the set $\mathcal{A}$ are measured from the origin O_v. Assume further that $\mathcal{B}$ is the set (or a subset) of all position vectors in the 3D space and they are measured from the origin O_w.

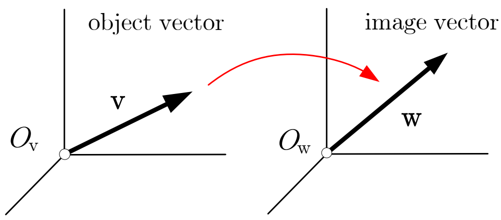

FIGURE 1.9. Mapping of $\mathbf{v}$ onto $\mathbf{w}$

Then we say that the vector-vector function

$$\mathbf{w} = \boldsymbol{\phi}(\mathbf{v}) \tag{1.55}$$

is a mapping of the vectors $\mathbf{v}$ (called object vectors) onto the vectors $\mathbf{w}$ (called image vectors). In other words: the vector-vector function $\mathbf{w} = \boldsymbol{\phi}(\mathbf{v})$ maps the 3D space onto itself (or a part of itself). In the second case the mapping is not one to one: it is called degenerated mapping.

The vector-vector function $\mathbf{w} = \boldsymbol{\phi}(\mathbf{v})$ is a homogeneous linear function if the functional equation

$$\boxed{\begin{aligned}\mathbf{w} = \boldsymbol{\phi}(\mathbf{v}) = \boldsymbol{\phi}(v_1\mathbf{i}_1 + v_2\mathbf{i}_2 + v_3\mathbf{i}_3) = v_1\underbrace{\boldsymbol{\phi}(\mathbf{i}_1)}_{\mathbf{w}_1} + v_2\underbrace{\boldsymbol{\phi}(\mathbf{i}_2)}_{\mathbf{w}_2} + v_3\underbrace{\boldsymbol{\phi}(\mathbf{i}_3)}_{\mathbf{w}_3} = \\ = \mathbf{w}_1 v_1 + \mathbf{w}_2 v_2 + \mathbf{w}_3 v_3\end{aligned}} \tag{1.56}$$

is satisfied where $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are the images of the base vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$.

Let us introduce the matrices of the vectors $\mathbf{w}$, $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$:

$$\underset{(3\times1)}{\underline{\mathbf{w}}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \quad \underset{(3\times1)}{\underline{\mathbf{w}_1}} = \begin{bmatrix} w_{11} \\ w_{21} \\ w_{31} \end{bmatrix}, \quad \underset{(3\times1)}{\underline{\mathbf{w}_2}} = \begin{bmatrix} w_{12} \\ w_{22} \\ w_{32} \end{bmatrix}, \quad \underset{(3\times1)}{\underline{\mathbf{w}_3}} = \begin{bmatrix} w_{13} \\ w_{23} \\ w_{33} \end{bmatrix}. \tag{1.57}$$

Making use of the matrix notations introduced we can rewrite (1.56) into the following form

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} w_{11} \\ w_{21} \\ w_{31} \end{bmatrix} v_1 + \begin{bmatrix} w_{12} \\ w_{22} \\ w_{32} \end{bmatrix} v_2 + \begin{bmatrix} w_{13} \\ w_{23} \\ w_{33} \end{bmatrix} v_3 \tag{1.58a}$$

or

$$\boxed{\begin{aligned}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} &= \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \\ \underset{(3\times1)}{\underline{\mathbf{w}}} &= \underbrace{\left[\ \underset{(3\times1)}{\underline{\mathbf{w}_1}}\ \middle|\ \underset{(3\times1)}{\underline{\mathbf{w}_2}}\ \middle|\ \underset{(3\times1)}{\underline{\mathbf{w}_3}}\ \right]}_{\underset{(3\times3)}{\underline{\mathbf{W}}}} \underset{(3\times1)}{\underline{\mathbf{v}}} = \underset{(3\times3)}{\underline{\mathbf{W}}}\,\underset{(3\times1)}{\underline{\mathbf{v}}}\end{aligned}} \tag{1.58b}$$

or

$$\boxed{w_k = w_{k\ell} v_\ell\ .} \tag{1.58c}$$

The nine scalar quantities $w_{k\ell}$ (the three image vectors $\mathbf{w}_\ell$) completely characterize (determine) the homogeneous linear mapping. The scalars $w_{k\ell}$ are referred to as the scalar components of a tensor of order two (or simply the components of a tensor). The concept of a tensor of order two will be detailed in the next subsection.

1.4.2.2. *Tensor products.* The tensor product (or dyadic)

$$\boxed{\boldsymbol{A} = \mathbf{a} \circ \mathbf{b}} \tag{1.59}$$

of the two vectors $\mathbf{a}$ and $\mathbf{b}$ is a tensor of order two $\boldsymbol{A}$ that assigns the vector $\mathbf{y}$ to each vector $\mathbf{x}$:

$$\boxed{\begin{aligned}\mathbf{y} &= (\mathbf{a}\circ\mathbf{b})\cdot\mathbf{x} = \mathbf{a}(\mathbf{b}\cdot\mathbf{x})\,,\\ y_\ell &= a_\ell(b_k x_k) = a_\ell b_k x_k\,.\end{aligned}} \tag{1.60}$$

REMARK 1.5: Equation (1.59) shows only the notation of the tensor (dyadic) product. To clarify the meaning of the product we have to give what rules are to be applied when various operations are performed on the product. In this respect equation (1.60) is of fundamental importance.

It follows from equation (1.60) that the tensor product is not commutative:

$$\mathbf{a}\circ\mathbf{b}\neq\mathbf{b}\circ\mathbf{a}\,. \tag{1.61}$$

In accordance with the definition given above (α is a scalar, $\mathbf{c}$ and $\mathbf{d}$ are vectors) it holds that

$$\underbrace{\mathbf{a}\circ\mathbf{b}}_{\boldsymbol{A}}+\underbrace{\mathbf{c}\circ\mathbf{d}}_{\boldsymbol{C}}=\boldsymbol{C}+\boldsymbol{A}\,, \qquad \text{(commutativity)} \tag{1.62a}$$

$$\mathbf{a}\circ(\mathbf{b}+\mathbf{c})=\mathbf{a}\circ\mathbf{b}+\mathbf{a}\circ\mathbf{c}\,, \qquad \text{(distributivity)} \tag{1.62b}$$

$$(\alpha\mathbf{a})\circ\mathbf{b}=\mathbf{a}\circ(\alpha\mathbf{b})=(\alpha\mathbf{a}\circ\mathbf{b})\;, \qquad \text{(associativity)}\,. \tag{1.62c}$$

The tensor product (or the sum of tensor products) is called tensor of order two (or simply tensor).

1.4.2.3. *Operations on tensors of order two.* The most important operations on tensors of order two (and their properties) are detailed in this subsection. It holds that

$$\left.\begin{aligned}\boldsymbol{A}\cdot\mathbf{c}=(\mathbf{a}\circ\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\,(\mathbf{b}\cdot\mathbf{c})\;,\\ \mathbf{c}\cdot\boldsymbol{A}=\mathbf{c}\cdot(\mathbf{a}\circ\mathbf{b})=(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}\,,\\ \boldsymbol{A}\cdot\mathbf{c}\neq\mathbf{c}\cdot\boldsymbol{A}\end{aligned}\right\} \tag{1.63}$$

or in indicial notation

$$\left.\begin{aligned}(\mathbf{a}\circ\mathbf{b})\cdot\mathbf{c}&=(a_k\mathbf{i}_k\circ b_\ell\mathbf{i}_\ell)\cdot c_r\mathbf{i}_r=a_k\mathbf{i}_k\,b_\ell c_r\underbrace{\mathbf{i}_\ell\cdot\mathbf{i}_r}_{\delta_{\ell r}}=\mathbf{i}_k a_k b_\ell c_\ell\,,\\ \mathbf{c}\cdot(\mathbf{a}\circ\mathbf{b})&=c_r\mathbf{i}_r\cdot(a_k\mathbf{i}_k\circ b_\ell\mathbf{i}_\ell)=\underbrace{\mathbf{i}_r\cdot\mathbf{i}_k}_{\delta_{rk}}c_r a_k b_\ell\mathbf{i}_\ell=c_r a_r b_\ell\mathbf{i}_\ell\,.\end{aligned}\right\} \tag{1.64}$$

REMARK 1.6: With a reference to Remark 1.3 we can omit the base vectors in the two scalar products. Then $a_k\,b_\ell c_\ell$ is the first and $c_r a_r\,b_\ell$ is the second scalar product.

Let $\mathbf{e}$ be a vector. It also holds that

$$\left.\begin{aligned}\mathbf{a}\circ\mathbf{b}\cdot(\mathbf{c}+\mathbf{d})=\mathbf{a}\,(\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{d})\;,\quad(\mathbf{c}+\mathbf{d})\cdot\mathbf{a}\circ\mathbf{b}=(\mathbf{c}\cdot\mathbf{a}+\mathbf{d}\cdot\mathbf{a})\,\mathbf{b}\,,\\ (\mathbf{a}\circ\mathbf{b}+\mathbf{c}\circ\mathbf{d})\cdot\mathbf{e}=\mathbf{a}\,(\mathbf{b}\cdot\mathbf{e})+\mathbf{c}\,(\mathbf{d}\cdot\mathbf{e})\;,\\ \mathbf{e}\cdot(\mathbf{a}\circ\mathbf{b}+\mathbf{c}\circ\mathbf{d})=(\mathbf{e}\cdot\mathbf{a})\,\mathbf{b}+(\mathbf{e}\cdot\mathbf{c})\,\mathbf{d}\,,\\ [(\alpha\mathbf{a})\circ\mathbf{b}]\cdot\mathbf{c}=[\mathbf{a}\circ(\alpha\mathbf{b})]\cdot\mathbf{c}=(\mathbf{a}\circ\mathbf{b})\cdot(\alpha\mathbf{c})\;,\\ (\alpha\mathbf{a})\cdot[\mathbf{b}\circ\mathbf{c}]=\mathbf{a}\cdot[(\alpha\mathbf{b})\circ\mathbf{c}]=\mathbf{a}\cdot[\mathbf{b}\circ(\alpha\mathbf{c})]\;.\end{aligned}\right\} \tag{1.65}$$

If we dot multiply $\boldsymbol{A} = \mathbf{a} \circ \mathbf{b}$ by $\mathbf{c}$ from left and by $\mathbf{d}$ from right we get

$$\mathbf{c} \cdot \boldsymbol{A} \cdot \mathbf{d} = \mathbf{c} \cdot (\mathbf{a} \circ \mathbf{b}) \cdot \mathbf{d} = (\mathbf{c} \cdot \mathbf{a})\,(\mathbf{b} \cdot \mathbf{d})\ . \tag{1.66}$$

As regards the cross product of a tensor and a vector the fundamental rules are as follows:

$$\begin{aligned} \boldsymbol{A} \times \boldsymbol{c} &= (\mathbf{a} \circ \mathbf{b}) \times \mathbf{c} = \mathbf{a} \circ (\mathbf{b} \times \mathbf{c})\ , \\ \boldsymbol{c} \times \boldsymbol{A} &= \boldsymbol{c} \times (\mathbf{a} \circ \mathbf{b}) = (\boldsymbol{c} \times \mathbf{a}) \circ \mathbf{b}\,, \\ \boldsymbol{c} \times \boldsymbol{A} \times \boldsymbol{d} &= \boldsymbol{c} \times (\mathbf{a} \circ \mathbf{b}) \times \boldsymbol{d} = (\boldsymbol{c} \times \mathbf{a}) \circ (\mathbf{b} \times \boldsymbol{d})\ . \end{aligned} \tag{1.67}$$

The result is always a tensor.

1.4.2.4. *Direct notation. Transformation rules.* Recalling equation (1.56) we can write

$$\mathbf{w} = \boldsymbol{\phi}(\mathbf{v}) = \mathbf{w}_1 v_1 + \mathbf{w}_2 v_2 + \mathbf{w}_3 v_3 = \mathbf{w}_1 \underbrace{(\mathbf{i}_1 \cdot \mathbf{v})}_{v_1} + \mathbf{w}_2 \underbrace{(\mathbf{i}_2 \cdot \mathbf{v})}_{v_2} + \mathbf{w}_3 \underbrace{(\mathbf{i}_3 \cdot \mathbf{v})}_{v_3}. \tag{1.68}$$

We can now factor out the vector $\mathbf{v}$ if we take rule $(1.60)_1$ into account. We get

$$\mathbf{w} = (\mathbf{w}_1 \circ \mathbf{i}_1) \cdot \mathbf{v} + (\mathbf{w}_2 \circ \mathbf{i}_2) \cdot \mathbf{v} + (\mathbf{w}_3 \circ \mathbf{i}_3) \cdot \mathbf{v} = \underbrace{(\mathbf{w}_\ell \circ \mathbf{i}_\ell)}_{\boldsymbol{W}} \cdot \mathbf{v} = \boldsymbol{W} \cdot \mathbf{v} \tag{1.69a}$$

or simply

$$\mathbf{w} = \boldsymbol{W} \cdot \mathbf{v}\,, \tag{1.69b}$$

where

$$\mathbf{W} = \mathbf{w}_\ell \circ \mathbf{i}_\ell = \underset{\mathbf{w}_\ell = w_{k\ell} \mathbf{i}_k}{\uparrow} = w_{k\ell}\, \mathbf{i}_k \circ \mathbf{i}_\ell \tag{1.70}$$

is the tensor $\boldsymbol{W}$ in terms of the tensor (dyadic) product $\mathbf{i}_k \circ \mathbf{i}_\ell$ which is called base tensor. It is obvious that

$$\mathbf{i}_m \cdot \boldsymbol{W} \cdot \mathbf{i}_n = \underset{(1.70)}{\uparrow} = w_{k\ell} \underbrace{(\mathbf{i}_m \cdot \mathbf{i}_k)}_{\delta_{mk}} \underbrace{(\mathbf{i}_\ell \cdot \mathbf{i}_n)}_{\delta_{\ell n}} = w_{k\ell}\, \delta_{mk}\, \delta_{\ell n} = w_{mn} \tag{1.71}$$

or briefly

$$\boxed{w_{mn} = \mathbf{i}_m \cdot \boldsymbol{W} \cdot \mathbf{i}_n\ .} \tag{1.72}$$

The tensor $\boldsymbol{W}$ characterizes the mapping, i.e., the homogeneous and linear vector-vector function $\mathbf{w} = \boldsymbol{\phi}(\mathbf{v})$ and should, therefore, be independent of the coordinate system applied. The tensor components w_{mn} and $w'_{k\ell}$ in the unprimed and primed coordinate systems depend, however, on the coordinate system we have selected. Consequently, it also holds

$$w'_{k\ell} = \mathbf{i}'_k \cdot \boldsymbol{W} \cdot \mathbf{i}'_\ell = \mathbf{i}'_k \cdot (w_{mn}\, \mathbf{i}_m \circ \mathbf{i}_n) \cdot \mathbf{i}'_\ell = \underset{(1.26b)}{\uparrow} = \underbrace{(\mathbf{i}'_k \cdot \mathbf{i}_m)}_{Q_{k'm}} \underbrace{(\mathbf{i}'_\ell \cdot \mathbf{i}_n)}_{Q_{\ell' n}} w_{mn} \tag{1.73a}$$

or briefly

$$\boxed{w'_{k\ell} = Q_{k'm}\, Q_{\ell' n}\, w_{mn}\ .} \tag{1.73b}$$

Hence, the tensor $\boldsymbol{W}$ in the primed coordinate system is given by the equation

$$\boldsymbol{W}' = w'_{k\ell}\, \mathbf{i}'_k \circ \mathbf{i}'_\ell\,. \tag{1.73c}$$

It follows from equations (1.73b) and (1.209) that the nine-nine scalars w_{mn} and $w'_{k\ell}$ constitute a tensor of order two if the two relations we have just referred to are satisfied.

REMARK 1.7: In accordance with equation (1.58b) the matrix of the tensor $\boldsymbol{W}$ is denoted either by $\underset{(3\times 3)}{\underline{\mathbf{W}}} = \underline{\mathbf{W}}$ or by $[w_{k\ell}]$.

REMARK 1.8: Assume that we know the vectors $\mathbf{g}_\ell$ ($[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] = \gamma_0 \neq 0$), which constitute a basis in the 3D space, and the dual base vectors $\mathbf{g}^k = \mathbf{g}^*_k$ as well. The tensor $\boldsymbol{W}$ maps the vector $\mathbf{g}_k$ onto $\hat{\mathbf{w}}_k$. With these image vectors the tensor $\boldsymbol{W}$ can be given in the following form:

$$\boldsymbol{W} = \hat{\mathbf{w}}_k \circ \mathbf{g}^*_k = \hat{\mathbf{w}}_1 \circ \mathbf{g}^*_1 + \hat{\mathbf{w}}_2 \circ \mathbf{g}^*_2 + \hat{\mathbf{w}}_3 \circ \mathbf{g}^*_3 \tag{1.74}$$

since this tensor maps the vectors $\mathbf{g}_\ell$ really onto the vectors $\hat{\mathbf{w}}_\ell$:

$$\boldsymbol{W} \cdot \mathbf{g}_\ell = (\hat{\mathbf{w}}_k \circ \mathbf{g}^*_k) \cdot \mathbf{g}_\ell = \hat{\mathbf{w}}_k\,(\mathbf{g}^*_k \cdot \mathbf{g}_\ell) = \hat{\mathbf{w}}_k \delta_{k\ell} = \hat{\mathbf{w}}_\ell\,.$$

Here we have taken the relation

$$\mathbf{g}^*_k \cdot \mathbf{g}_\ell = \delta_{k\ell}\,, \tag{1.75}$$

which follows from a comparison of (1.22) and (1.37), into account.

1.4.3. Special tensors.

1.4.3.1. *Identitiy tensor and zero tensor.* We denote the identity tensor by $\boldsymbol{1}$ the zero tensor by $\boldsymbol{O}$. They are defined by the equations

$$\boxed{\underbrace{\boldsymbol{1} \cdot \mathbf{v} = \mathbf{v}}_{\delta_{k\ell} v_\ell = v_k}, \qquad \boldsymbol{O} \cdot \mathbf{v} = \mathbf{0}\,.} \tag{1.76}$$

In words: the identity tensor (or unit tensor) maps a vector $\mathbf{v}$ onto itself; the zero tensor maps a vector $\mathbf{v}$ onto the zero vector. It is clear that

$$\begin{gathered} \boldsymbol{1} = \mathbf{i}_k \circ \mathbf{i}_k = \mathbf{i}'_k \circ \mathbf{i}'_k \\ (\boldsymbol{1} \cdot \mathbf{v} = (\mathbf{i}_k \circ \mathbf{i}_k) \cdot \mathbf{v} = \mathbf{i}_k(\mathbf{i}_k \cdot \mathbf{v}) = v_k \mathbf{i}_k = \mathbf{v}) \end{gathered}\;; \quad \underset{(3\times 3)}{\underline{\mathbf{1}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underset{(3\times 3)}{\underline{\mathbf{O}}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.77}$$

REMARK 1.9: Assume again that we know the vectors $\mathbf{g}_\ell$ ($[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3] = \gamma_0 \neq 0$) and the dual base vectors $\mathbf{g}^k = \mathbf{g}^*_k$ as well. Then the identity tensor (unit tensor) can also be given in terms of these base vectors as

$$\boxed{\boldsymbol{1} = \mathbf{g}_k \circ \mathbf{g}^*_k = \mathbf{g}_1 \circ \mathbf{g}^*_1 + \mathbf{g}_2 \circ \mathbf{g}^*_2 + \mathbf{g}_3 \circ \mathbf{g}^*_3} \tag{1.78}$$

since

$$\boldsymbol{1}\cdot\mathbf{v}=(\mathbf{g}_k\circ\mathbf{g}_k^*)\cdot\mathbf{v}=\mathbf{g}_k\,(\mathbf{g}_k^*\cdot\mathbf{v})=\underset{(1.24)}{\uparrow}=\mathbf{v},$$

which shows that the tensor $\boldsymbol{1}=\mathbf{g}_k\circ\mathbf{g}_k^*$ maps $\mathbf{v}$ really onto itself.

1.4.3.2. *Dot product of two tensors.* The dot product of two tensors – they are denoted by $\boldsymbol{S}$ and $\boldsymbol{T}$, the product by $\boldsymbol{U}$ – is defined by the composition

$$\boldsymbol{S}\cdot\boldsymbol{T}\cdot\mathbf{v}=\boldsymbol{S}\cdot(\boldsymbol{T}\cdot\mathbf{v})\ ,\quad \boldsymbol{S}\cdot\boldsymbol{T}=\boldsymbol{U} \tag{1.79}$$

which should hold for every vector $\mathbf{v}$. Here

$$\boldsymbol{T}\cdot\mathbf{v}=t_{k\ell}\,\mathbf{i}_k\circ\mathbf{i}_\ell\cdot v_m\mathbf{i}_m=t_{k\ell}\,\mathbf{i}_k\underbrace{(\mathbf{i}_\ell\cdot\mathbf{i}_m)}_{\delta_{\ell m}}v_m=\mathbf{i}_k\,t_{k\ell}\,\delta_{\ell m}v_m=\mathbf{i}_k\,t_{k\ell}\,v_\ell\,.$$

Hence,

$$\boldsymbol{U}\cdot\mathbf{v}=\boldsymbol{S}\cdot(\boldsymbol{T}\cdot\mathbf{v})=(s_{pq}\,\mathbf{i}_p\circ\mathbf{i}_q)\cdot\mathbf{i}_k\,t_{k\ell}\,v_\ell=s_{pq}\,\mathbf{i}_p\underbrace{(\mathbf{i}_q\cdot\mathbf{i}_k)}_{\delta_{qk}}t_{k\ell}\,v_\ell=\mathbf{i}_p\,\underbrace{s_{pq}t_{q\ell}}_{u_{p\ell}}\,v_\ell,$$

which means that

$$\boxed{\boldsymbol{U}=\boldsymbol{S}\cdot\boldsymbol{T}=s_{pq}t_{q\ell}\,\mathbf{i}_p\circ\mathbf{i}_\ell\ ,\qquad u_{p\ell}=s_{pq}t_{q\ell}\ ,} \tag{1.80a}$$

or in matrix notation

$$\boxed{\underset{(3\times3)}{\underline{\mathbf{U}}}=\underset{(3\times3)}{\underline{\mathbf{S}}}\,\underset{(3\times3)}{\underline{\mathbf{T}}}\ .} \tag{1.80b}$$

1.4.3.3. *Transposition.* The transpose of a tensor – say the tensor $\boldsymbol{S}$ – is denoted by $\boldsymbol{S}^T$. For the tensor $\boldsymbol{S}=s_{pq}\,\mathbf{i}_p\circ\mathbf{i}_q$ the transpose is defined by the equation

$$\boxed{\boldsymbol{S}^T=(s^T)_{pq}\,\mathbf{i}_p\circ\mathbf{i}_q=s_{pq}\,\mathbf{i}_q\circ\mathbf{i}_p\,.} \tag{1.81a}$$

This means that the transpose is obtained by interchanging the order of the two factors in the dyadic product. It is obvious that the definition given above is valid for any tensor of order two in the Cartesian coordinate system we use. If we rename p to q and q to p on the right side of (1.81a) we get

$$\boldsymbol{S}^T=\underline{(s^T)_{pq}}\,\mathbf{i}_p\circ\mathbf{i}_q=\underline{s_{qp}}\,\mathbf{i}_p\circ\mathbf{i}_q\,, \tag{1.81b}$$

in which the dyadic products are the same. Hence

$$\boxed{(s^T)_{pq}=s_{qp}\,.} \tag{1.81c}$$

Since the first subscript counts the rows while the second the columns in the matrix of a tensor we obtain that the rows in the matrix of $\boldsymbol{S}^T$ are the same as the columns in the matrix of $\boldsymbol{S}$. It follows from (1.81c) that

$$\underset{(3\times3)}{\underline{\mathbf{S}}^T}=\Big(\underset{(3\times3)}{\underline{\mathbf{S}}}\Big)^T\,. \tag{1.82}$$

In words: the matrix of $\boldsymbol{S}^T$ is equal to the transpose of the matrix of $\boldsymbol{S}$.

Let α and β two scalars. Further let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. The most important properties and relations concerning the transposition are:

$$\begin{gathered}
\left(\boldsymbol{S}^T\right)^T = \boldsymbol{S}\,, \\
(\alpha \boldsymbol{S} + \beta \boldsymbol{W})^T = \alpha \boldsymbol{S}^T + \beta \boldsymbol{W}^T\,, \\
(\mathbf{a} \circ \mathbf{b})^T = \mathbf{b} \circ \mathbf{a}\,, \\
\boldsymbol{S} \cdot \mathbf{u} = \mathbf{u} \cdot \boldsymbol{S}^T\,, \quad \mathbf{u} \cdot \boldsymbol{S} = \boldsymbol{S}^T \cdot \mathbf{u}\,, \quad \mathbf{u} \cdot \boldsymbol{S} \cdot \mathbf{v} = \mathbf{v} \cdot \boldsymbol{S}^T \cdot \mathbf{u}\,, \\
(\boldsymbol{S} \cdot \boldsymbol{W})^T = s_{k\ell} w_{\ell r} \mathbf{i}_r \circ \mathbf{i}_k = w_{pr} \mathbf{i}_r \circ \overbrace{\mathbf{i}_p \cdot \mathbf{i}_\ell}^{\delta_{p\ell}} \circ \mathbf{i}_k \, s_{k\ell} = \boldsymbol{W}^T \cdot \boldsymbol{S}^T.
\end{gathered} \tag{1.83}$$

A definition of the transpose which is independent of the coordinate system is as follows: $\boldsymbol{S}^T$ is the unique tensor which satisfies the relation

$$\mathbf{u} \cdot \boldsymbol{S} \cdot \mathbf{v} = \mathbf{v} \cdot \boldsymbol{S}^T \cdot \mathbf{u} \tag{1.84}$$

for any $\mathbf{u}$ and $\mathbf{v}$

1.4.3.4. *Symmetric and skew tensors.* A tensor, say the tensor $\boldsymbol{S}$, is said to be symmetric if

$$\boldsymbol{S} = \boldsymbol{S}^T, \quad \left(\text{then } \mathbf{i}_k \cdot \boldsymbol{S} \cdot \mathbf{i}_\ell = \mathbf{i}_\ell \cdot \boldsymbol{S}^T \cdot \mathbf{i}_k = \mathbf{i}_\ell \cdot \boldsymbol{S} \cdot \mathbf{i}_k \quad \longrightarrow \quad s_{k\ell} = s_{\ell k}\right). \tag{1.85a}$$

A tensor, say the tensor $\boldsymbol{S}$, is said to be skew if

$$\boldsymbol{S} = -\boldsymbol{S}^T, \quad \left(\text{then } \mathbf{i}_k \cdot \boldsymbol{S} \cdot \mathbf{i}_\ell = \mathbf{i}_\ell \cdot \boldsymbol{S}^T \cdot \mathbf{i}_k = -\mathbf{i}_\ell \cdot \boldsymbol{S} \cdot \mathbf{i}_k \quad \longrightarrow \quad s_{k\ell} = -s_{\ell k}\right). \tag{1.85b}$$

If $\boldsymbol{S}$ is skew

$$s_{11} = s_{22} = s_{33} = 0\,. \tag{1.85c}$$

On the basis of equation (1.84) we can also say that the tensor $\boldsymbol{S}$ is [symmetric]{skew} if for any $\mathbf{u}$ and $\mathbf{v}$ the following relation is satisfied:

$$[\mathbf{u} \cdot \boldsymbol{S} \cdot \mathbf{v} = \mathbf{v} \cdot \boldsymbol{S} \cdot \mathbf{u}] \ \ \{\mathbf{u} \cdot \boldsymbol{S} \cdot \mathbf{v} = -\mathbf{v} \cdot \boldsymbol{S} \cdot \mathbf{u}\}\,. \tag{1.86}$$

1.4.3.5. *The trace of a tensor.* Equation

$$\boxed{\operatorname{tr}(\boldsymbol{S}) = s_{k\ell}\, \mathbf{i}_k \cdot \mathbf{i}_\ell = s_{k\ell} \delta_{k\ell} = s_{\ell\ell} = s_{11} + s_{22} + s_{33}} \tag{1.87}$$

defines the trace of the tensor $\boldsymbol{S}$.

1.4.3.6. *Additive resolution.* Making use of the identity

$$\boxed{\boldsymbol{S} = \underbrace{\frac{1}{2}\left(\boldsymbol{S} + \boldsymbol{S}^T\right)}_{\boldsymbol{S}_{\text{sym}}} + \underbrace{\frac{1}{2}\left(\boldsymbol{S} - \boldsymbol{S}^T\right)}_{\boldsymbol{S}_{\text{skew}}} = \boldsymbol{S}_{\text{sym}} + \boldsymbol{S}_{\text{skew}}} \tag{1.88}$$

we define the symmetric and skew parts of the tensor $\boldsymbol{S}$ by the following relations

$$\boxed{\begin{aligned}
\boldsymbol{S}_{\text{sym}} &= \frac{1}{2}\left(\boldsymbol{S} + \boldsymbol{S}^T\right), \quad & \boldsymbol{S}_{\text{skew}} &= \frac{1}{2}\left(\boldsymbol{S} - \boldsymbol{S}^T\right), \\
s_{(k\ell)} &= \frac{1}{2}\left(s_{k\ell} + s_{\ell k}\right), & s_{[k\ell]} &= \frac{1}{2}\left(s_{k\ell} - s_{\ell k}\right).
\end{aligned}} \tag{1.89}$$

Equation (1.89) is the additve resolution of the tensor $\boldsymbol{S}$ into symmetric and skew parts.

1.4.3.7. *Axial vector.* For the mapping that belongs to $\boldsymbol{S}_{\text{skew}}$ we can write – $\mathbf{n}$ is the vector we map (the object vector) – that

$$\boldsymbol{S}_{\text{skew}}\cdot\mathbf{n}=\frac{1}{2}\left(s_{pq}\,\mathbf{i}_p\circ\mathbf{i}_q-s_{pq}\,\mathbf{i}_q\circ\mathbf{i}_p\right)\cdot n_v\mathbf{i}_v=$$
$$=-\frac{1}{2}s_{pq}\left[\mathbf{i}_q\left(\mathbf{i}_p\cdot\mathbf{i}_v\right)-\mathbf{i}_p\left(\mathbf{i}_q\cdot\mathbf{i}_v\right)\right]n_v=\underset{(1.16)}{\uparrow}=\underbrace{-\frac{1}{2}s_{pq}\left(\mathbf{i}_p\times\mathbf{i}_q\right)}_{\boldsymbol{s}^a}\times\underbrace{\mathbf{i}_v n_v}_{\mathbf{n}}=\boldsymbol{s}^a\times\boldsymbol{n}\,. \tag{1.90}$$

Here

$$\boxed{\begin{gathered}\boldsymbol{s}^a=-\frac{1}{2}\,s_{pq}\left(\mathbf{i}_p\times\mathbf{i}_q\right)=-\frac{1}{2}\,e_{pqr}\,s_{pq}\,\mathbf{i}_r\,,\\ s^{(a)}_r=-\frac{1}{2}\,e_{pqr}\,s_{pq}\end{gathered}} \tag{1.91}$$

is called axial vector. Since the mapping

$$\boxed{\boldsymbol{S}_{\text{skew}}\cdot\mathbf{n}=\boldsymbol{s}^a\times\boldsymbol{n}} \tag{1.92}$$

is independent of the coordinate system applied it follows that the axial vector $\boldsymbol{s}^a$ is also independent of the coordinate system. For this reason $\boldsymbol{s}^a$ is often referred to as vector invariant.

EXERCISE 1.5: Assume that we know $s^{(a)}_r$. Prove that

$$s_{[k\ell]}=-e_{k\ell r}s^{(a)}_r\,. \tag{1.93}$$

Multiplying throughout equation (1.91)$_2$ by $-e_{k\ell r}$ we get

$$-e_{k\ell r}s^{(a)}_r=\frac{1}{2}\underbrace{e_{k\ell r}e_{pqr}}_{\delta_{kp}\delta_{\ell q}-\delta_{kq}\delta_{\ell p}}s_{pq}=\frac{1}{2}\left(\delta_{kp}\delta_{\ell q}s_{pq}-\delta_{kq}\delta_{\ell p}s_{pq}\right)\frac{1}{2}\left(s_{k\ell}-s_{\ell k}\right)=s_{[k\ell]}\,.$$

That was to be proved.

EXERCISE 1.6: Assume that $\boldsymbol{S}$ is a symmetric tensor. Show that then its axial vector is zero vector.

Making use of (1.91)$_2$ and omitting the terms in which the permutation symbol is equal to zero we can write

$$\begin{aligned}
s^{(a)}_1&=-\frac{1}{2}s_{pq}\,e_{pq1}=-\frac{1}{2}\big(s_{23}\underbrace{e_{231}}_{=1}+s_{32}\underbrace{e_{321}}_{=-1}\big)=-\frac{1}{2}\big(s_{23}-s_{32}\big)=0\,,\\
s^{(a)}_2&=-\frac{1}{2}s_{pq}\,e_{pq2}=-\frac{1}{2}\big(s_{31}\underbrace{e_{312}}_{=1}+s_{13}\underbrace{e_{132}}_{=-1}\big)=-\frac{1}{2}\big(s_{31}-s_{13}\big)=0\,,\\
s^{(a)}_3&=-\frac{1}{2}s_{pq}\,e_{pq3}=-\frac{1}{2}\big(s_{12}\underbrace{e_{123}}_{=1}+s_{21}\underbrace{e_{213}}_{=-1}\big)=-\frac{1}{2}\big(s_{12}-s_{21}\big)=0\,.
\end{aligned} \tag{1.94}$$

It follows from this result that the axial vector of a tensor is equal to the axial vector of its skew part. It is also obvious that a tensor with zero axial vector is a symmetric tensor, i.e., the vanishing axial vector is a symmetry condition.

1.4.3.8. *Inner (or energy) product.* Let $\boldsymbol{S} = s_{pq}\mathbf{i}_p \circ \mathbf{i}_q$ and $\boldsymbol{T} = t_{rs}\mathbf{i}_r \circ \mathbf{i}_s$ be two tensors. The inner or double dot product of the two tensors is defined by the following equation:

$$\boxed{\begin{aligned}\boldsymbol{S} \cdot\cdot \boldsymbol{T} &= \operatorname{tr}\left(\boldsymbol{S}^T \cdot \boldsymbol{T}\right) = \operatorname{tr}\big(s_{pq}\mathbf{i}_q \circ \underbrace{\mathbf{i}_p \cdot \mathbf{i}_r}_{\delta_{pr}} \circ \mathbf{i}_s t_{rs}\big) = \\ &= \operatorname{tr}\big(s_{rq}\mathbf{i}_q \circ \mathbf{i}_s t_{rs}\big) = s_{rq}\underbrace{\mathbf{i}_q \cdot \mathbf{i}_s}_{\delta_{qs}} t_{rs} = s_{rq}t_{rq} = t_{rq}s_{rq} = \boldsymbol{T} \cdot\cdot \boldsymbol{S}\,.\end{aligned}} \tag{1.95}$$

The inner product has, among others, the following properties:

$$\begin{aligned}\boldsymbol{1} \cdot\cdot \boldsymbol{1} &= \delta_{rs}\delta_{rs} = \delta_{rr} = 3\,, \\ \boldsymbol{T} \cdot\cdot \boldsymbol{1} &= t_{rs}\delta_{rs} = t_{rr} = t_{11} + t_{22} + t_{33}\,, \\ \boldsymbol{T} \cdot\cdot (\mathbf{v} \circ \mathbf{u}) &= t_{rs}\, v_r u_s = v_r\, t_{rs}\, u_s = \mathbf{v} \cdot \boldsymbol{T} \cdot \mathbf{u}\,, \\ (\mathbf{v} \circ \mathbf{u}) \cdot\cdot (\mathbf{t} \circ \mathbf{w}) &= v_r u_s\, t_r w_s = v_r t_r\, u_s w_s = (\mathbf{v} \cdot \mathbf{t})\,(\mathbf{u} \cdot \mathbf{w})\;.\end{aligned} \tag{1.96}$$

Let $\boldsymbol{T} = t_{rs}\mathbf{i}_r \circ \mathbf{i}_s$ be symmetric and $\boldsymbol{S} = s_{pq}\mathbf{i}_p \circ \mathbf{i}_q$ be skew. Then

$$\begin{aligned}\boldsymbol{T} \cdot\cdot \boldsymbol{S} = t_{rq}s_{rq} &= \underset{s_{11}=s_{22}=s_{33}=0}{\uparrow} \quad \underset{t_{12}=t_{21},\; t_{23}=t_{32},\; t_{31}=t_{13}}{\uparrow} = \\ &= t_{12}\underbrace{(s_{12} + s_{21})}_{=0} + t_{23}\underbrace{(s_{23} + s_{32})}_{=0} + t_{31}\underbrace{(s_{31} + s_{13})}_{=0} = 0\,.\end{aligned} \tag{1.97}$$

In words: The inner product of a symmetric and a skew tensor is equal to zero.

1.4.3.9. *Determinant and inverse.* The determinant of a tensor $\boldsymbol{S}$ regarded in the Cartesian coordinate system (x_1, x_2, x_3) is the determinant of its matrix:

$$\det(\boldsymbol{S}) = \det(\underset{(3\times 3)}{\underline{\mathbf{S}}}) = |s_{k\ell}| = \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix}. \tag{1.98}$$

The determinant has, among others, the following properties:

$$\begin{aligned}\det(\boldsymbol{S}) &= \det(\boldsymbol{S}^T)\,, \\ \det(\boldsymbol{S} \cdot \boldsymbol{T}) &= \det(\boldsymbol{S})\det(\boldsymbol{T})\,.\end{aligned} \tag{1.99}$$

According to equation $(1.99)_2$ the determinant of the product $\boldsymbol{S} \cdot \boldsymbol{T}$ is equal to the product of the determinants $\det(\boldsymbol{S})$ and $\det(\boldsymbol{T})$.

The inverse of a tensor $\boldsymbol{S}$ is the tensor $\boldsymbol{S}^{-1}$ for which

$$\boxed{\boldsymbol{S} \cdot \boldsymbol{S}^{-1} = \boldsymbol{S}^{-1} \cdot \boldsymbol{S} = \boldsymbol{1}\;.} \tag{1.100}$$

Assume that $\det(\boldsymbol{S}) \neq 0$. Then the image vectors $\mathbf{s}_k = \boldsymbol{S} \cdot \mathbf{i}_k$ constitute a basis in the 3D space. It is obvious from Subsection 1.1.7 that the dual base vectors that belong to the image vectors $\mathbf{s}_k$ are given by

$$\begin{aligned}\mathbf{s}^*_1 = \frac{\mathbf{s}_2 \times \mathbf{s}_3}{s_\circ}\,, \qquad \mathbf{s}^*_2 &= \frac{\mathbf{s}_3 \times \mathbf{s}_1}{s_\circ}\,, \qquad \mathbf{s}^*_3 = \frac{\mathbf{s}_1 \times \mathbf{s}_2}{s_\circ}\,, \\ s_\circ = [\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3] = \det(\boldsymbol{S}) &\neq 0\,, \qquad \mathbf{s}^*_k \cdot \mathbf{s}_\ell = \delta_{k\ell}\,.\end{aligned} \tag{1.101}$$

With the dual base vectors

$$\boxed{\boldsymbol{S}^{-1} = \mathbf{i}_k \circ \mathbf{s}_k^*} \tag{1.102}$$

is the inverse of the tensor $\boldsymbol{S}$ since

$$\boldsymbol{S}^{-1} \cdot \boldsymbol{S} = (\mathbf{i}_k \circ \mathbf{s}_k^*) \cdot (\mathbf{s}_\ell \circ \mathbf{i}_\ell) = \mathbf{i}_k \circ \mathbf{i}_\ell \underbrace{(\mathbf{s}_k^* \cdot \mathbf{s}_\ell)}_{\delta_{k\ell}} = \mathbf{i}_\ell \circ \mathbf{i}_\ell = \boldsymbol{1} \,.$$

The tensor has an inverse if its determinant is not zero. Then the tensor is non-singular. The inverse has the following properties:

$$\begin{aligned} \left(\boldsymbol{S}^{-1}\right)^{-1} &= \boldsymbol{S} \,, \\ (\alpha \boldsymbol{S})^{-1} &= \frac{1}{\alpha}\boldsymbol{S}^{-1} \,, \\ \left(\boldsymbol{S}^{-1}\right)^{T} &= \left(\boldsymbol{S}^{T}\right)^{-1} \,, \\ \det(\boldsymbol{S}^{-1}) &= \frac{1}{\det(\boldsymbol{S})} \,, \\ (\boldsymbol{S} \cdot \boldsymbol{T})^{-1} &= \boldsymbol{T}^{-1} \cdot \boldsymbol{S}^{-1} \,. \end{aligned} \tag{1.103}$$

In the sequel we shall apply, for the sake of simplicity, the following notational convention [83]:

$$\boxed{\boldsymbol{S}^{-T} = \left(\boldsymbol{S}^{-1}\right)^{T} = \left(\boldsymbol{S}^{T}\right)^{-1} .} \tag{1.104}$$

The tensor components $\mathcal{S}_{p\ell}$ are defined by the following equation

$$\mathcal{S}_{p\ell} = \frac{1}{2} e_{pqr} e_{\ell jk} s_{jq} s_{kr} \,. \tag{1.105}$$

With $\mathcal{S}_{p\ell}$ it holds that

$$\begin{aligned} s_{ip}\mathcal{S}_{p\ell} = s_{ip}\frac{1}{2} e_{pqr} e_{\ell jk} s_{jq} s_{kr} \underset{(1.47)}{=\uparrow=} e_{\ell jk}\frac{1}{2} \underbrace{e_{pqr} s_{ip} s_{jq} s_{kr}}_{e_{ijk}|s_{mn}|} = \\ \underset{(1.49)_2}{=\uparrow=} \underbrace{e_{ijk} e_{\ell jk}}_{2\delta_{i\ell}} \frac{1}{2} |s_{mn}| = \delta_{i\ell} |s_{mn}| \,. \end{aligned}$$

Consequently, the components of the inverse $\boldsymbol{S}^{-1}$ can determined by using the following relationship:

$$s_{p\ell}^{-1} = \frac{\mathcal{S}_{p\ell}}{|s_{mn}|} \,. \tag{1.106}$$

The matrix constituted by the tensor components $\mathcal{S}_{p\ell}$ is called adjugate matrix.

1.4.3.10. *Integer powers.* The integer powers of a tensor $\boldsymbol{S}$ are defined by the equation

$$\boldsymbol{S}^n = \boldsymbol{S}^{n-1} \cdot \boldsymbol{S} \,, \qquad n \geq 1 \tag{1.107a}$$

where n is a positive integer and

$$\boldsymbol{S}^0 = \boldsymbol{1} \,. \tag{1.107b}$$

1.4.3.11. *A categorization.* Let $\mathbf{v}$ be an arbitrary non zero vector. A tensor $\boldsymbol{W}$ is said to be

$$\boxed{\begin{array}{lcl} \text{positive definite} & & > 0 \\ \text{positive semidefinite} & \text{if the product} \quad \mathbf{v}\cdot\boldsymbol{W}\cdot\mathbf{v} & \geq 0 \\ \text{negative semidefinite} & & \leq 0 \\ \text{negative definite} & & < 0 \end{array}} \tag{1.108}$$

for any $\mathbf{v}$ and it is indefinite if none of the above inequalities is satisfied.

1.4.4. Eigenvalue problem of a symmetric tensor.

1.4.4.1. *Principal directions and characteristic equation.* Let $\boldsymbol{W}$ be a symmetric tensor. We seek those directions (denoted in general by n and called principal directions) for which the object vector $\mathbf{n}$ and the image vector $\mathbf{w}_n$ are

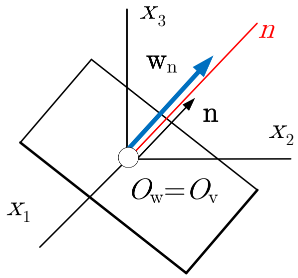

FIGURE 1.10. Eigenvalue problem

parallel to each other, i.e., it holds

$$\mathbf{w}_n = \boldsymbol{W}\cdot\mathbf{n} = \lambda\mathbf{n}\,; \qquad |\mathbf{n}| = 1\,, \tag{1.109}$$

where the eigenvector $\mathbf{n} = n_\ell \mathbf{i}\ell$ (under the side condition $|\mathbf{n}| = 1$) and the parameter λ (called eigenvalue or principal value) are the unknowns. Since the unit tensor $\boldsymbol{1}$ maps a vector onto itself we can rewrite condition $(1.109)_1$ into the following form:

$$\boxed{\begin{array}{c} (\boldsymbol{W} - \lambda\boldsymbol{1})\cdot\mathbf{n} = \mathbf{0}\,, \\ (w_{k\ell} - \lambda\delta_{k\ell})\,n_\ell = 0\,. \end{array}} \tag{1.110}$$

This equation is, in fact, a homogeneous linear equation system for the unknowns n_ℓ:

$$\underbrace{\begin{bmatrix} w_{11}-\lambda & w_{12} & w_{13} \\ w_{21} & w_{22}-\lambda & w_{23} \\ w_{31} & w_{32} & w_{33}-\lambda \end{bmatrix}}_{\underline{\mathbf{W}}-\lambda\underline{\mathbf{1}}} \underbrace{\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}}_{\underline{\mathbf{n}}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{1.111}$$

The trivial solution for n_ℓ does not satisfy the side condition $|\mathbf{n}| = 1$. Hence a solution different from the trivial one exists if and only if

$$P_3(\lambda) = -\det(\underline{\mathbf{W}} - \lambda\underline{\mathbf{1}}) = -[(\mathbf{w}_1 - \lambda\mathbf{i}_1)(\mathbf{w}_2 - \lambda\mathbf{i}_2)(\mathbf{w}_3 - \lambda\mathbf{i}_3)] = \\ = -|w_{k\ell} - \lambda\delta_{k\ell}| = 0\,. \quad (1.112a)$$

This equation is a cubic polynomial of λ:

$$P_3(\lambda) = -\begin{vmatrix} w_{11}-\lambda & w_{12} & w_{13} \\ w_{21} & w_{22}-\lambda & w_{23} \\ w_{31} & w_{32} & w_{33}-\lambda \end{vmatrix} = \\ = \lambda^3 - W_I\lambda^2 + W_{II}\lambda - W_{III} = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = 0\,, \quad (1.112b)$$

in which

$$W_I = [\mathbf{w}_1\mathbf{i}_2\mathbf{i}_3] + [\mathbf{i}_1\mathbf{w}_2\mathbf{i}_3] + [\mathbf{i}_1\mathbf{i}_2\mathbf{w}_3] = w_{11} + w_{22} + w_{33} = w_{\ell\ell}\,, \quad (1.113a)$$

$$W_{II} = [\mathbf{w}_1\mathbf{w}_2\mathbf{i}_3] + [\mathbf{w}_1\mathbf{i}_2\mathbf{w}_3] + [\mathbf{i}_1\mathbf{w}_2\mathbf{w}_3] = \\ = \begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} + \begin{vmatrix} w_{11} & w_{13} \\ w_{31} & w_{33} \end{vmatrix} + \begin{vmatrix} w_{22} & w_{23} \\ w_{32} & w_{33} \end{vmatrix} = \\ = \frac{1}{2}(w_{kk}w_{\ell\ell} - w_{k\ell}w_{\ell k}) = \frac{1}{2}\left(W_I^2 - w_{k\ell}w_{\ell k}\right)\,, \quad (1.113b)$$

$$W_{III} = [\mathbf{w}_1\mathbf{w}_2\mathbf{w}_3] = |w_{k\ell}| = \begin{vmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{vmatrix} = e_{k\ell r}w_{1k}w_{2\ell}w_{3r}\,. \quad (1.113c)$$

and λ_1, λ_2 and λ_3 are the roots of the polynomial $P_3(\lambda) = 0$ – Viéte's theorem states that there are three roots[3] [1].

It is also clear on the basis of equation (1.112b) that

$$W_I = \lambda_1 + \lambda_2 + \lambda_3\,, \quad W_{II} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\,, \quad W_{III} = \lambda_1\lambda_2\lambda_3\,. \quad (1.114)$$

The resolvability condition $P_3(\lambda) = 0$ is called characteristic equation which can be solved for the unknown parameter λ.

The coefficients W_I, W_{II} and W_{III} in the characteristic equation are referred to as scalar invariants. Since the problem raised by equation (1.109) is independent of the coordinate system it follows that the solutions for λ (the roots of the polynomial $P_3(\lambda) = 0$) should also be independent of the coordinate system. Hence the coefficients in the characteristic equation $P_3(\lambda) = 0$ should also be coordinate system independent quantities, therefore they are really invariants.

It can be proved that the roots λ_ℓ are real if $\boldsymbol{W}$ is symmetric – this is assumption – and are positive numbers if $\boldsymbol{W}$ is positive definite. The roots are labeled in such manner that $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

Problem (1.109) is an eigenvalue problem, the roots λ_ℓ are the eigenvalues, the solutions for $\mathbf{n}_\ell$ are called eigenvectors, while the directions that belong to the eigenvectors are the principal directions.

If the roots λ_ℓ are different there exist three different eigenvectors (one to each root) which are mutually perpendicular to each other. If two roots (or all the three) coincide then the number of eigenvectors is infinite, however one can

[3] Francois Viéte (1540-1603)

always select such three which are mutually perpendicular to each other. In the sequel it is assumed that the eigenvectors are mutually perpendicular to each other.

1.4.4.2. *Spectral decomposition.* It follows from all that has been said above that the principal directions can constitute a right handed Cartesian coordinate system for which the eigenvectors $\mathbf{n}_\ell$ are the base vectors. Hence

$$\boldsymbol{1} = \mathbf{n}_\ell \circ \mathbf{n}_\ell \tag{1.115}$$

is the unit tensor in this coordinate system. Equation

$$\boxed{\boldsymbol{W} = \boldsymbol{W} \cdot \boldsymbol{1} = \underbrace{\boldsymbol{W} \cdot \mathbf{n}_\ell}_{\lambda_\ell \mathbf{n}_\ell} \circ \mathbf{n}_\ell = \sum_{\ell=1}^{3} \lambda_\ell \, \mathbf{n}_\ell \circ \mathbf{n}_\ell} \tag{1.116}$$

is the spectral decomposition of the tensor $\boldsymbol{W}$. Its components in the coordinate system of the principal directions $(n_1 n_2 n_3)$ are given by

$$\underset{(n)}{w_{k\ell}} = \mathbf{n}_k \cdot \underbrace{\boldsymbol{W} \cdot \mathbf{n}_\ell}_{\lambda_\ell \mathbf{n}_\ell} = \lambda_\ell \underbrace{\mathbf{n}_k \cdot \mathbf{n}_\ell}_{\delta_{k\ell}} = \lambda_\ell \delta_{k\ell} \, , \qquad \text{(no sum on } \ell). \tag{1.117}$$

Consequently, $\boldsymbol{W}$ has a diagonal matrix in the coordinate system of the principal directions:

$$\underset{(n)}{\underline{\mathbf{W}}} = [\underset{(n)}{w_{k\ell}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} . \tag{1.118}$$

Let $\lambda_1 = \lambda_2$ be a double root. Then λ_3 and the third principal direction, i.e., $\mathbf{n}_3$ is uniquely determined while $\mathbf{n}_1$ and $\mathbf{n}_2$ lie in a plane perpendicular to $\mathbf{n}_3$. Though they can be arbitrary it is worth choosing them in such a way that the orthogonality condition $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$ be satisfied. The dyadic form of $\boldsymbol{W}$ can now be obtained by taking the equality $\lambda_1 = \lambda_2$ into account. We get

$$\begin{aligned} \boldsymbol{W} &= \lambda_1 \left(\mathbf{n}_1 \circ \mathbf{n}_1 + \mathbf{n}_2 \circ \mathbf{n}_2\right) + \lambda_3 \mathbf{n}_3 \circ \mathbf{n}_3 = \\ &= \lambda_1 \underbrace{\left(\mathbf{n}_1 \circ \mathbf{n}_1 + \mathbf{n}_2 \circ \mathbf{n}_2 + \mathbf{n}_3 \circ \mathbf{n}_3\right)}_{\boldsymbol{1}} + (\lambda_3 - \lambda_1)\, \mathbf{n}_3 \circ \mathbf{n}_3 = \lambda_1 \boldsymbol{1} + (\lambda_3 - \lambda_1)\, \mathbf{n}_3 \circ \mathbf{n}_3 \, . \end{aligned} \tag{1.119}$$

If we have a triple root $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ it follows from (1.119) that

$$\boldsymbol{W} = \lambda \boldsymbol{1} \, . \tag{1.120}$$

Consequently every direction is a principal direction. This tensor is called spherical tensor since it maps a sphere onto a sphere.

The inverse of the tensor $\boldsymbol{W}$ in the coordinate system of the principal axes is given by

$$\boldsymbol{W}^{-1} = \sum_{\ell=1}^{3} \frac{1}{\lambda_\ell} \mathbf{n}_\ell \circ \mathbf{n}_\ell \tag{1.121}$$

This can be checked with ease if we take into account (1.115) when we calculate the product $\boldsymbol{W}^{-1} \cdot \boldsymbol{W}$. Equations (1.116) and (1.121) show that a tensor and its inverse are coaxial, i.e., their principal directions are the same.

Let

$$\mathbf{n} = n^1\mathbf{g}_1 + n^2\mathbf{g}_2 + n^3\mathbf{g}_3\,,\, |\mathbf{n}| = 1 \tag{1.122}$$

be the eigenvector in the basis constituted by the vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$. The dual base vectors are given by (1.21). Remembering that the tensor $\boldsymbol{W}$ and the unit tensor $\boldsymbol{1}$ in this basis are given by (1.74) and (1.78) we may rewrite polynomial (1.111) into the following form

$$P_3(\lambda) = -\det(\underline{\mathbf{W}} - \lambda\underline{\mathbf{1}}) = -\left[(\hat{\mathbf{w}}_1 - \lambda\mathbf{g}_1)\,(\hat{\mathbf{w}}_2 - \lambda\mathbf{g}_2)\,(\hat{\mathbf{w}}_3 - \lambda\mathbf{g}_3)\right]. \tag{1.123}$$

Hence,

$$\begin{aligned} P_3(\lambda) = {} & [\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]\,\lambda^3 - \left([\hat{\mathbf{w}}_1\mathbf{g}_2\mathbf{g}_3] + [\mathbf{g}_1\hat{\mathbf{w}}_2\mathbf{g}_3] + [\mathbf{g}_1\mathbf{g}_2\hat{\mathbf{w}}_3]\right)\lambda^2 + \\ & + \left([\hat{\mathbf{w}}_1\hat{\mathbf{w}}_2\mathbf{g}_3] + [\hat{\mathbf{w}}_1\mathbf{g}_2\hat{\mathbf{w}}_3] + [\mathbf{g}_1\hat{\mathbf{w}}_2\hat{\mathbf{w}}_3]\right)\lambda - [\hat{\mathbf{w}}_1\hat{\mathbf{w}}_2\hat{\mathbf{w}}_3] = 0 \end{aligned} \tag{1.124}$$

is the characteristic equation. A comparison of (1.124) and (1.112b) yields the three scalar invariants in the basis formed by the vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ [69, p. 1986]:

$$\begin{aligned} W_I = {} & \frac{[(\boldsymbol{W}\cdot\mathbf{g}_1)\,\mathbf{g}_2\,\mathbf{g}_3] + [\mathbf{g}_1\,(\boldsymbol{W}\cdot\mathbf{g}_2)\,\mathbf{g}_3] + [\mathbf{g}_1\,\mathbf{g}_2\,(\boldsymbol{W}\cdot\mathbf{g}_3)]}{[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]} = \\ & = \mathbf{g}_1^*\cdot\boldsymbol{W}\cdot\mathbf{g}_1 + \mathbf{g}_2^*\cdot\boldsymbol{W}\cdot\mathbf{g}_2 + \mathbf{g}_3^*\cdot\boldsymbol{W}\cdot\mathbf{g}_3\,, \end{aligned} \tag{1.125a}$$

$$W_{II} = \frac{[(\boldsymbol{W}\cdot\mathbf{g}_1)(\boldsymbol{W}\cdot\mathbf{g}_2)\mathbf{g}_3] + [(\boldsymbol{W}\cdot\mathbf{g}_1)\mathbf{g}_2(\boldsymbol{W}\cdot\mathbf{g}_3)] + [\mathbf{g}_1(\boldsymbol{W}\cdot\mathbf{g}_2)(\boldsymbol{W}\cdot\mathbf{g}_3)]}{[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]}\,, \tag{1.125b}$$

$$W_{III} = \frac{[(\boldsymbol{W}\cdot\mathbf{g}_1)\,(\boldsymbol{W}\cdot\mathbf{g}_2)\,(\boldsymbol{W}\cdot\mathbf{g}_3)]}{[\mathbf{g}_1\mathbf{g}_2\mathbf{g}_3]}\,, \tag{1.125c}$$

where we have taken the properties of the box product and relation (1.21) also into account.

1.4.4.3. *Square root and logarithm.* Let the symmetric tensor $\boldsymbol{W}$ be positive definite. Then the eigenvalues meet the condition $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. The square root of the tensor $\boldsymbol{W}$ is defined by the following equation:

$$\boxed{\sqrt{\boldsymbol{W}} = \sum_{\ell=1}^{3} \sqrt{\lambda_\ell}\,\mathbf{n}_\ell \circ \mathbf{n}_\ell\,.} \tag{1.126}$$

The square root of a positive definite symmetric tensor is a unique mathematical operation. Elegant proofs of this statement are presented in [76, 60].

It can be checked with ease that $\sqrt{\boldsymbol{W}}\cdot\sqrt{\boldsymbol{W}} = \boldsymbol{W}$.

The definition of the natural logarithm is given by the equation

$$\boxed{\ln\boldsymbol{W} = \sum_{\ell=1}^{3} (\ln\lambda_\ell)\;\mathbf{n}_\ell \circ \mathbf{n}_\ell\,.} \tag{1.127}$$

Note that we need the spectral decomposition for calculating the square root and the natural logarithm of a positive definite symmetric tensor. It also follows from the definition that a positive definite symmetric tensor has one square root only.

1.4.4.4. *Cayley-Hamilton theorem.* The Cayley-Hamilton theorem [4] states that a tensor satisfies its characteristic equation [17], [15] – see book [75] for a detailed proof:

$$\boxed{\boldsymbol{W}^3 - W_I \boldsymbol{W}^2 + W_{II} \boldsymbol{W} - W_{III} \boldsymbol{1} = \boldsymbol{O}\,.} \tag{1.128}$$

This equation can be used to reduce a tensor polynomial of higher order to a tensor polynomial of lower order.

If we dot multiply equation (1.128) by $\boldsymbol{W}^{-1}$ we get the following representation for its inverse:

$$\boxed{\boldsymbol{W}^{-1} = \frac{1}{W_{III}} \left(\boldsymbol{W}^2 - W_I \boldsymbol{W} + W_{II} \boldsymbol{1} \right)\,.} \tag{1.129}$$

If we take the double dot product of (1.128) and the identity tensor and then utilize $(1.96)_{1,2}$ we get a new relationship for the third invariant. In the first step we have

$$\begin{aligned} \boldsymbol{W}^3 \cdot\cdot\, \boldsymbol{1} - W_I \boldsymbol{W}^2 \cdot\cdot\, \boldsymbol{1} + W_{II} \boldsymbol{W} \cdot\cdot \boldsymbol{1} - W_{III} \boldsymbol{1} \cdot\cdot\, \boldsymbol{1} = \\ = w_{kp} w_{pq} w_{qk} - W_I w_{k\ell} w_{\ell k} + W_{II} W_I - 3 W_{III} = 0 \end{aligned}$$

from where it follows that

$$W_{III} = \frac{1}{3} \left(w_{kp} w_{pq} w_{qk} - W_I w_{k\ell} w_{\ell k} + W_{II} W_I \right)\,.$$

Substitute now (1.113b) first for W_{II} and then for the product $w_{k\ell} w_{\ell k}$. After a rearrangement we get

$$W_{III} = \frac{1}{6} \left(-2 W_I^3 + 6 W_I W_{II} + 2 w_{kp} w_{pq} w_{qk} \right)\,. \tag{1.130}$$

1.4.4.5. *Coaxial tensors.* Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two symmetric tensors: $\boldsymbol{A} = \boldsymbol{A}^T$, $\boldsymbol{B} = \boldsymbol{B}^T$. They are said to be coaxial if their principal directions coincide. Let us denote the eigenvalues and the corresponding principal directions for tensor $\boldsymbol{A}$ by χ_ℓ and $\mathbf{n}_\ell$, $|\mathbf{n}_\ell| = 1$. Assume that

$$\boldsymbol{B} = \alpha \boldsymbol{A} + \beta \mathbf{1}\,.$$

in which α and β are non zero scalars. Then

$$\boldsymbol{B} \cdot \mathbf{n}_\ell = (\alpha \boldsymbol{A} + \beta \mathbf{1}) \cdot \mathbf{n}_\ell = \alpha \underbrace{\boldsymbol{A} \cdot \mathbf{n}_\ell}_{\chi_\ell \mathbf{n}_\ell} + \beta \underbrace{\mathbf{1} \cdot \mathbf{n}_\ell}_{\mathbf{n}_\ell} = \underbrace{(\alpha \chi_\ell + \beta)\, \mathbf{n}_\ell}_{\text{parallel to } n_\ell}, \quad \text{(no sum on } \ell)$$

where $\alpha \chi_\ell + \beta$ is the eigenvalue of $\boldsymbol{B}$. This equation shows that the tensor $\boldsymbol{B}$ defined above and the tensor $\boldsymbol{A}$ are coaxial.

If the tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ are coaxial, then

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$$

which shows that the dot product is a commutative operation for coaxial tensors.

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[4] Arthur Cayley (1821–1895), William Roman Hamilton (1805-1865)

EXERCISE 1.7: Given the matrix of a tensor $\boldsymbol{T}$:

$$\underline{\mathbf{T}} = \begin{bmatrix} 85 & 0 & 25 \\ 0 & -10 & 0 \\ 25 & 0 & -35 \end{bmatrix} \qquad [\mathrm{N/mm^2}]$$

Determine the eigenvalues and eigenvectors.

It is obvious that $\mathbf{i}_2$ is a principal direction with the eigenvalue $\lambda^{(a)} = -10$ [N/mm^2] – this is the second element in the second column of the matrix. After substitutions the characteristic equation

$$P_3(\lambda) = -\det(\underline{\mathbf{T}} - \lambda \underline{\mathbf{1}}) = -\begin{vmatrix} t_{11}-\lambda & t_{12} & t_{13} \\ t_{21} & t_{22}-\lambda & t_{23} \\ t_{31} & t_{32} & t_{33}-\lambda \end{vmatrix} =$$
$$= \lambda^3 - T_I\lambda^2 + T_{II}\lambda - T_{III} = (\lambda - \lambda^{(a)})(\lambda - \lambda^{(b)})(\lambda - \lambda^{(c)}) = 0$$

yields

$$P_3(\lambda) = \begin{vmatrix} 85-\lambda & 0 & 25 \\ 0 & -10-\lambda & 0 \\ 25 & 0 & -35-\lambda \end{vmatrix} = \lambda^3 - 40\lambda^2 - 4100\lambda - 36\,000 = 0$$

in which

$$T_I = \lambda^{(a)} + \lambda^{(b)} + \lambda^{(c)} = 40\,, \quad T_{II} = -4100\,, \quad T_{III} = \lambda^{(a)}\lambda^{(b)}\lambda^{(c)} = 36\,000\,.$$

Here the scalar invariants T_I and T_{III} are given in the coordinate system of the principal directions – since we do not know the ordered set of the eigenvalues they are denoted simply by $\lambda^{(a)}$, $\lambda^{(b)}$ and $\lambda^{(c)}$. If $\lambda \neq \lambda^{(a)}$ we can divide the characteristic equation $P_3(\lambda) = 0$ by $\lambda - \lambda^{(a)}$. We get

$$\frac{P_3(\lambda)}{\lambda - \lambda^{(a)}} = (\lambda - \lambda^{(b)})(\lambda - \lambda^{(c)}) = \lambda^2 - (\lambda^{(b)} + \lambda^{(c)})\lambda + \lambda^{(b)}\lambda^{(c)} = 0\,,$$

where

$$\lambda^{(b)} + \lambda^{(c)} = T_I - \lambda^{(a)} = 50 \qquad \text{and} \qquad \lambda^{(b)}\lambda^{(c)} = \frac{T_{III}}{\lambda^{(a)}} = -36\,00\,.$$

Hence, the solutions of equation

$$\lambda^2 - (T_I - \lambda^{(a)})\lambda + \frac{T_{III}}{\lambda^{(a)}} = \lambda^2 - 50\lambda - 36\,00 = 0$$

for λ result in the two missing eigenvalues: $\lambda^{(b)} = 90$, $\lambda^{(c)} = -40$. We can now give the ordered set of the eigenvalues:

$$\lambda_1 = \lambda^{(b)} = 90\,, \qquad \lambda_2 = \lambda^{(a)} = -10\,, \qquad \lambda_3 = \lambda^{(c)} = -40\,.$$

To find $\mathbf{n}_1 = n_{11}\mathbf{i}_1 + n_{21}\mathbf{i}_2 + n_{31}\mathbf{i}_3$ we have to solve the equation system

$$\begin{bmatrix} t_{11}-\lambda_1 & t_{12} & t_{13} \\ t_{21} & t_{22}-\lambda_1 & t_{23} \\ t_{31} & t_{32} & t_{33}-\lambda_1 \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix} =$$

$$= \begin{bmatrix} 85-\lambda_1 & 0 & 25 \\ 0 & -10-\lambda_1 & 0 \\ 25 & 0 & -35-\lambda_1 \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

from where we get

$$\begin{aligned} -5n_{11} + 25n_{31} &= 0\,, & n_{11} &= 5n_{31}\,, \\ -100n_{21} &= 0\,, & n_{21} &= 0\,, \\ 25n_{11} - 125n_{31} &= 0\,, & n_{11} &= 5n_{31}\,. \end{aligned} \tag{1.131}$$

Equations $(1.131)_1$ and $(1.131)_3$ are not independent. It follows from equations $(1.131)_{1,2}$ and the side condition $|\mathbf{n}_1| = 1$ that

$$\mathbf{n}_1 = \frac{1}{\sqrt{26}}(5\mathbf{i}_1 + \mathbf{i}_3)\,.$$

If we take into account that the eigenvectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ constitute a right handed orthonormal basis we get:

$$\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2 = \frac{1}{\sqrt{26}}(5\mathbf{i}_1 + \mathbf{i}_3) \times \mathbf{i}_2 = \frac{1}{\sqrt{26}}(-\mathbf{i}_1 + 5\mathbf{i}_3)\,.$$

1.4.5. Orthogonal tensors. Let $\boldsymbol{Q}$ be an invertible tensor of order two. Further let $\mathbf{p}$ and $\mathbf{s}$ be the image vectors of the object vectors $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{p} = \boldsymbol{Q} \cdot \mathbf{v}\,, \qquad \mathbf{s} = \boldsymbol{Q} \cdot \mathbf{w}\,. \tag{1.132}$$

The tensor $\boldsymbol{Q}$ is orthogonal if it holds for any $\mathbf{v}$ and $\mathbf{w}$ that

$$\mathbf{p} \cdot \mathbf{s} = (\boldsymbol{Q} \cdot \mathbf{v}) \cdot (\boldsymbol{Q} \cdot \mathbf{w}) = \mathbf{v} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}\,. \tag{1.133}$$

After rearranging the above equation we have

$$\mathbf{v} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot \left(\boldsymbol{Q}^T \cdot \boldsymbol{Q} - \boldsymbol{1}\right) \cdot \mathbf{w} = \mathbf{0}\,, \tag{1.134}$$

in which $\mathbf{v}$ and $\mathbf{w}$ are arbitrary. Hence

$$\boldsymbol{Q}^T \cdot \boldsymbol{Q} = \boldsymbol{1} \tag{1.135}$$

from where it also follows that

$$\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}\,. \tag{1.136}$$

In words: the transpose of an orthogonal tensor coincides with its inverse.

Assume that $\mathbf{v} = \mathbf{w}$. Then $\mathbf{p} = \mathbf{s}$ and equation (1.133) yields

$$\mathbf{p} \cdot \mathbf{s} = \mathbf{s}^2 = \mathbf{w} \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q} \cdot \mathbf{w} = \mathbf{w}^2\,,$$

which shows that the lengths of the object vector $\mathbf{w}$ and image vector $\mathbf{s}$ are the same, mapping (1.132) preserves the distance.

Let the angles formed by the vectors $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{p}$, $\mathbf{s}$ be denoted by ϑ and φ – $\vartheta, \varphi \in [0, \pi]$. Recalling the definition of the dot product – see equation (1.4) – we can write on the basis of (1.133) that

$$\mathbf{p} \cdot \mathbf{s} = |\mathbf{p}|\,|\mathbf{s}| \cos\varphi = |\mathbf{v}|\,|\mathbf{w}| \cos\vartheta = \mathbf{v} \cdot \mathbf{w}\,,$$

where

$$|\mathbf{p}| = |\mathbf{v}| \quad \text{and} \quad |\mathbf{s}| = |\mathbf{w}|\,,$$

thus

$$\cos\varphi = \cos\vartheta$$

or

$$\varphi = \vartheta\,. \tag{1.137}$$

Consequently, mapping (1.132) preserves the angles as well.

Consider now the determinant of the product $\boldsymbol{Q}^T \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{Q}^T = \boldsymbol{1}$. Making use of equation $(1.99)_2$ we have

$$\det\big(\boldsymbol{Q}^T \cdot \boldsymbol{Q}\big) = \det\big(\boldsymbol{Q}^T\big)\det(\boldsymbol{Q}) = [\det(\boldsymbol{Q})]^2 = \det(\boldsymbol{1}) = 1\,,$$

from where

$$\det(\boldsymbol{Q}) = \pm 1\,. \tag{1.138}$$

Examine now the issue if there exists such a vector (denoted by $\mathbf{s}$) for which

$$\boldsymbol{Q} \cdot \mathbf{s} = \pm\mathbf{s}\,. \tag{1.139}$$

If yes then

$$\boldsymbol{Q}^T \cdot \mathbf{s} = \pm\underbrace{\boldsymbol{Q}^T \cdot \boldsymbol{Q}}_{\boldsymbol{1}} \cdot \mathbf{s} = \pm\mathbf{s}\,. \tag{1.140}$$

Subtract (1.140) from (1.139) and divide the result by two. We get

$$\frac{1}{2}\big(\boldsymbol{Q} - \boldsymbol{Q}^T\big) \cdot \mathbf{s} = \boldsymbol{Q}_{\text{skew}} \cdot \mathbf{s} = \mathbf{0}\,.$$

Let us denote the axial vector of $\boldsymbol{Q}$ by $\mathbf{q}^a$. On the basis of (1.90) we can rewrite the previous equation:

$$\boldsymbol{Q}_{\text{skew}} \cdot \mathbf{s} = \mathbf{q}^a \times \mathbf{s} = \mathbf{0}\,. \tag{1.141}$$

This result means that the solution for $\mathbf{s}$ in equation (1.139) is parallel to the axial vector $\mathbf{q}^a$ of the tensor $\boldsymbol{Q}$.

Consider now the eigenvalue problem

$$\begin{gathered} \boldsymbol{Q} \cdot \mathbf{s} = \lambda\mathbf{s}\,, \qquad |\mathbf{s}| = 1 \\ [\lambda \text{ is the root of the polinomial} \det(\boldsymbol{Q} - \lambda\boldsymbol{1}) = 0] \end{gathered} \tag{1.142}$$

for the orthogonal tensor $\boldsymbol{Q}$. Note that this problem coincides with problem (1.139) we have raised above if $\lambda = \pm 1$.

If $\mathbf{s}$ is eigenvector and λ is eigenvalue then

$$\lambda^2 = \lambda^2\mathbf{s} \cdot \mathbf{s} = \lambda\mathbf{s} \cdot \lambda\mathbf{s} = (\boldsymbol{Q} \cdot \mathbf{s}) \cdot (\boldsymbol{Q} \cdot \mathbf{s}) = \mathbf{s} \cdot \underbrace{\boldsymbol{Q}^T \cdot \boldsymbol{Q}}_{\boldsymbol{1}} \cdot \mathbf{s} = \mathbf{s} \cdot \mathbf{s} = 1\,, \tag{1.143}$$

which shows that λ is really ± 1.

Let us clarify what role the signs play in the mapping.

Assume first that $\det(\boldsymbol{Q}) = 1$. In the following manipulations (a) we take into account that the determinants of a tensor and its transpose are the same, (b) we utilize equation (1.135) and (c) we apply the product theorem of the determinants:

$$\det(\boldsymbol{Q}-\boldsymbol{1}) = \det\left[(\boldsymbol{Q}-\boldsymbol{1})^T\right] = \det\left(\boldsymbol{Q}^T-\boldsymbol{1}\right) = \det\left(\boldsymbol{Q}^T-\boldsymbol{Q}^T\cdot\boldsymbol{Q}\right) = \\ = \underbrace{\det\left(\boldsymbol{Q}^T\right)}_{\det(\boldsymbol{Q})=1}\det(\boldsymbol{1}-\boldsymbol{Q}) = -\det(\boldsymbol{Q}-\boldsymbol{1})\,.$$

If a scalar is equal to its opposite then the scalar is zero. Hence,

$$\det(\boldsymbol{Q}-\lambda\boldsymbol{1})|_{\lambda=1} = \det(\boldsymbol{Q}-\boldsymbol{1}) = 0 \tag{1.144a}$$

which shows that

$$\lambda = 1 \qquad \text{if} \qquad \det(\boldsymbol{Q}) = 1\,. \tag{1.144b}$$

Assume now that $\det(\boldsymbol{Q}) = -1$ and examine the determinant $\det(\boldsymbol{Q}+\boldsymbol{1})$. By repeating the steps leading to (1.144b) we get

$$\det(\boldsymbol{Q}+\boldsymbol{1}) = \det\left[(\boldsymbol{Q}+\boldsymbol{1})^T\right] = \det\left(\boldsymbol{Q}^T+\boldsymbol{1}\right) = \det\left(\boldsymbol{Q}^T+\boldsymbol{Q}^T\cdot\boldsymbol{Q}\right) = \\ = \underbrace{\det\left(\boldsymbol{Q}^T\right)}_{\det(\boldsymbol{Q})=-1}\det(\boldsymbol{1}+\boldsymbol{Q}) = -\det(\boldsymbol{Q}+\boldsymbol{1})\,.$$

Hence,

$$\det(\boldsymbol{Q}-\lambda\boldsymbol{1})|_{\lambda=-1} = \det(\boldsymbol{Q}+\boldsymbol{1}) = 0 \tag{1.144c}$$

which shows that

$$\lambda = -1 \qquad \text{if} \qquad \det(\boldsymbol{Q}) = -1\,. \tag{1.144d}$$

It follows from what has been said above that

$$\begin{array}{lll} \boldsymbol{Q}\cdot\mathbf{q}^a = \mathbf{q}^a\,, & \text{if} \quad \det(\boldsymbol{Q}) = 1 & [\text{since then } \lambda = 1] \text{ and} \\ \boldsymbol{Q}\cdot\mathbf{q}^a = -\mathbf{q}^a\,, & \text{if} \quad \det(\boldsymbol{Q}) = -1 & [\text{since then } \lambda = -1]\,. \end{array} \tag{1.145}$$

Note that according to (1.140) it also holds that

$$\boldsymbol{Q}^T\cdot\mathbf{q}^a = \pm\mathbf{q}^a\,. \tag{1.146}$$

Making use of equations (1.145) and (1.146) we can clarify the geometric character of the mapping. The object vector $\mathbf{v}$ can be resolved into two components: one, parallel to $\mathbf{q}_a$, the other perpendicular to it:

$$\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp} \qquad \left(\mathbf{v}_{||}\times\mathbf{q}^a = \mathbf{0}\,, \qquad \mathbf{v}_{\perp}\cdot\mathbf{q}^a = 0\right)\,.$$

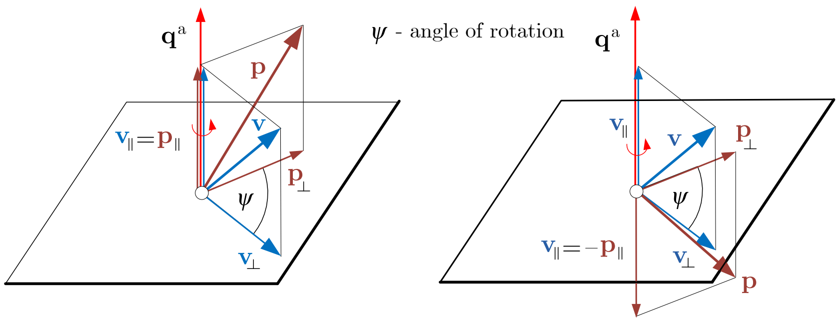

FIGURE 1.11. (a) Rotation (b) Rotation and reflection

Thus

$$\mathbf{p} = \boldsymbol{Q} \cdot \mathbf{v} = \underbrace{\boldsymbol{Q} \cdot \mathbf{v}_{||}}_{\mathbf{p}_{||}} + \underbrace{\boldsymbol{Q} \cdot \mathbf{v}_{\perp}}_{\mathbf{p}_{\perp}}$$

is the image vector.

Since the mapping is distance preserving and $\mathbf{v}_{||}$ is parallel to $\mathbf{q}^a$ it follows from (1.145) that

$$\begin{array}{lll} \mathbf{p}_{||} = \mathbf{v}_{||} & \text{which means that } \mathbf{p}_{||} \text{ coincides with } \mathbf{v}_{||} \text{ if} & \det(\boldsymbol{Q}) = 1\,, \\ \mathbf{p}_{||} = -\mathbf{v}_{||} & \text{which means that } \mathbf{p}_{||} \text{ is the reflection of } \mathbf{v}_{||} \text{ if} & \det(\boldsymbol{Q}) = -1\,. \end{array}$$

The image of $\mathbf{v}_{\perp}$, i.e., $\mathbf{p}_{\perp}$ is also perpendicular to $\mathbf{q}^a$. If we use (1.146) we may write

$$\mathbf{q}^a \cdot \mathbf{p}_{\perp} = \mathbf{q}^a \cdot \boldsymbol{Q} \cdot \mathbf{v}_{\perp} = \mathbf{v}_{\perp} \cdot \underbrace{\boldsymbol{Q}^T \cdot \mathbf{q}^a}_{\pm \mathbf{q}^a} = \pm \mathbf{q}^a \cdot \mathbf{v}_{\perp} = 0\,,$$

which shows that $\mathbf{v}_{\perp}$ rotates in the plane perpendicular to $\mathbf{q}^a$ – the angle of rotation is denoted by ψ.

Consequently, the mapping $\boldsymbol{Q} \cdot \mathbf{v}$ is a finite rotation if $\det(\boldsymbol{Q}) = 1$ and is a finite rotation plus reflection if $\det(\boldsymbol{Q}) = -1$.

If $\det(\boldsymbol{Q}) = 1$ the tensor $\boldsymbol{Q}$ is called proper orthogonal and is denoted by $\boldsymbol{R}$:

$$\det(\boldsymbol{R}) = 1\,, \qquad \boldsymbol{R} \cdot \boldsymbol{R}^T = \boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{1}\,. \tag{1.147}$$

Its name is rotation tensor since the mapping which belongs to $\boldsymbol{R}$ is finite rotation.

Let n and ψ be the rotation axis and the angle of rotation. The direction vector of the axis n is denoted by $\mathbf{n}||\mathbf{q}^a$ ($|\mathbf{n}| = 1$) – see Figure A.1 in the Appendix A for further details. If ($Q_{III} = 1$) [$Q_{III} = -1$] the mapping $\mathbf{p} = \boldsymbol{Q} \cdot \mathbf{v}$ that belongs to the orthogonal Rodrigues[5] tensor [2]

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{1}\, \cos\psi + (Q_{III} - \cos\psi)\, \mathbf{n} \circ \mathbf{n} + \boldsymbol{1} \times \mathbf{n}\, \sin\psi\,, \\ Q_{k\ell} &= \delta_{k\ell}\, \cos\psi + (Q_{III} - \cos\psi)\, n_k n_\ell + \delta_{kn} e_{nr\ell}\, n_r\, \sin\psi \end{aligned} \tag{1.148}$$

[5]Olinde Rodrigues, 1795-1851

(rotates the vector $\mathbf{v}$ about the axis n through the angle ψ) [rotates the vector $\mathbf{v}$ and reflects it with respect to a plane perpendicular to the axis n]. The proof of this statement is presented in Section A.1.2.

1.4.6. Tensors of higher order.

1.4.6.1. *Tensors of order zero, one and two.* A scalar field, say the temperature distribution in a body, is independent of the coordinate system we use to describe it. Because of this property scalars are called tensors of order zero. Vectors should behave as if they were position vectors which remain unchanged when we rotate the coordinate system about the origin, i.e., the vector components in the unprimed and primed coordinate systems should follow transformation rule (1.54). The concept of tensors of order two was introduced via a coordinate system independent mapping which resulted in that the tensor components $w'_{k\ell}$ and w_{mn} should satisfy transformation rule (1.73b). The first three rows in Table 1. show the properties (requirements) the scalars t and t',

TABLE 1.

	Tensor of order	Number of independent tensor components	Law of transformation	Called
1.	0	1	$t = t'$	Scalar
2.	1	3	$t_m = Q_{m\ell'} t'_\ell$	Vector
3.	2	9	$t_{mn} = Q_{mk'} Q_{n\ell'} t'_{k\ell}$	Tensor
4.	3	27	$t_{mnp} = Q_{mk'} Q_{n\ell'} Q_{pr'} t'_{k\ell r}$	Triad
5.	4	81	$t_{mnpq} = Q_{mk'} Q_{n\ell'} Q_{pr'} Q_{qs'} t'_{k\ell rs}$	Tetrad

the vector components t_m and t'_ℓ, as well as the tensor components t_{mn} and $t'_{k\ell}$ should meet.

1.4.6.2. *Tensors of order higher than two.* A generalization of the above mentioned requirements leads to a definition concerning the tensors of higher order: [t_{mnp} and $t'_{k\ell r}$] {t_{mnpq} and $t'_{k\ell rs}$} are the scalar components of a tensor of order [three] {four} if the equations (the transformation rules) in rows [4] and {5} of Table 1 are satisfied.

REMARK 1.10: Table 1 shows the short names of these quantities: a tensor of order zero is called a scalar, a tensor of order one is a vector, a tensor of order two is called simply tensor, a tensor of order three is a triad, a tensor of order four is a tetrad [85].

EXERCISE 1.8: Is the Kronecker delta a tensor of order two?
Yes it is. Equation

$$\delta_{mn} = \mathbf{i}_m \cdot \mathbf{i}_n = \underset{(1.29c)}{\uparrow} = Q_{mk'}\, \mathbf{i}'_k \cdot Q_{n\ell'}\, \mathbf{i}'_\ell = Q_{mk'}\, Q_{n\ell'}\, \underbrace{\mathbf{i}'_k \cdot \mathbf{i}'_\ell}_{\delta'_{k\ell}} = Q_{mk'}\, Q_{n\ell'}\, \delta'_{k\ell} \tag{1.149}$$

shows that the transformation rule the tensor components $\delta_{k\ell}$ and δ'_{mn} should meet is really satisfied.

By using symbolic (or direct) notation tensors of order three or four can be written in the forms:

$$ {}^{(3)}\boldsymbol{S} = s_{k\ell r}\,\mathbf{i}_k \circ \mathbf{i}_\ell \circ \mathbf{i}_r\,, \qquad {}^{(4)}\boldsymbol{C} = c_{k\ell rs}\,\mathbf{i}_k \circ \mathbf{i}_\ell \circ \mathbf{i}_r \circ \mathbf{i}_s\,. \tag{1.150} $$

It is a notational convention that the superscript (a number in parentheses) which precedes the letter that identifies the tensor shows the order of the tensor in question.

REMARK 1.11: We, in general, do not apply this convention to scalars, vectors and tensors of order two. The permutation tensor is also an exception to this rule since in direct notation it will be denoted by $\boldsymbol{\mathcal{E}}$. Thus

$$ \boldsymbol{\mathcal{E}} = e_{k\ell r}\,\mathbf{i}_k \circ \mathbf{i}_\ell \circ \mathbf{i}_r\,. \tag{1.151} $$

The outer product of a tensor ${}^{(r)}\boldsymbol{A}$ of order r and a tensor ${}^{(s)}\boldsymbol{B}$ of order s is a tensor ${}^{(r+s)}\boldsymbol{D}$ of order $r+s$. If $r=2$ and $s=3$ we have

$$ {}^{(5)}\boldsymbol{D} = {}^{(2)}\boldsymbol{A}\circ{}^{(3)}\boldsymbol{B} = \underbrace{a_{k\ell}\,b_{mnr}}_{d_{k\ell mnr}}\,\mathbf{i}_k\circ\mathbf{i}_\ell\circ\mathbf{i}_m\circ\mathbf{i}_n\circ\mathbf{i}_r = d_{k\ell mnr}\mathbf{i}_k\circ\mathbf{i}_\ell\circ\mathbf{i}_m\circ\mathbf{i}_n\circ\mathbf{i}_r\,. \tag{1.152a} $$

If we omit the base vectors we can simply write

$$ d_{k\ell mnr} = a_{k\ell}\,b_{mnr}\,. \tag{1.152b} $$

The dot product of a tensor ${}^{(r)}\boldsymbol{B}$ of order r and a tensor ${}^{(s)}\boldsymbol{C}$ of order s is a tensor ${}^{(r+s-2)}\boldsymbol{A}$ of order $r+s-2$ defined by the following relationship ($r=3$, $s=4$):

$$ {}^{(5)}\boldsymbol{A} = {}^{(3)}\boldsymbol{B}\cdot{}^{(4)}\boldsymbol{C} = b_{k\ell m}\,\mathbf{i}_k \circ \mathbf{i}_\ell \circ \underbrace{\mathbf{i}_m \cdot \mathbf{i}_p}_{\delta_{mp}} \circ \mathbf{i}_q \circ \mathbf{i}_r \circ \mathbf{i}_s\,c_{pqrs} = $$

$$ = \underbrace{b_{k\ell m}\,c_{mqrs}}_{a_{k\ell qrs}}\,\mathbf{i}_k \circ \mathbf{i}_\ell \circ \mathbf{i}_q \circ \mathbf{i}_r \circ \mathbf{i}_s = a_{k\ell qrs}\,\mathbf{i}_k \circ \mathbf{i}_\ell \circ \mathbf{i}_q \circ \mathbf{i}_r \circ \mathbf{i}_s\,. \tag{1.153a} $$

If we omit the base vectors we have

$$ a_{k\ell qrs} = b_{k\ell m}\,c_{mqrs}\,. \tag{1.153b} $$

Operations (1.152a) and (1.153a) are not commutative.

The double dot product of a tensor ${}^{(r)}\boldsymbol{A}$ of order r and a tensor ${}^{(s)}\boldsymbol{B}$ of order s is a tensor ${}^{(r+s-4)}\boldsymbol{D}$ of order $r+s-4$ defined by the following relationship ($r=3$, $s=4$):

$$ {}^{(3)}\boldsymbol{D} = {}^{(3)}\boldsymbol{A}\cdot\cdot{}^{(4)}\boldsymbol{B} = (a_{k\ell m}\mathbf{i}_k \circ \mathbf{i}_\ell \circ \mathbf{i}_m)\cdot\cdot(b_{pqrs}\mathbf{i}_p \circ \mathbf{i}_q \circ \mathbf{i}_r \circ \mathbf{i}_s) = $$

$$ = a_{k\ell m}\mathbf{i}_k \circ \underbrace{\boxed{\mathbf{i}_\ell \cdot \mathbf{i}_p}}_{\delta_{\ell p}}\,\underbrace{\boxed{\mathbf{i}_m \cdot \mathbf{i}_q}}_{\delta_{mq}}\,b_{pqrs}\mathbf{i}_r \circ \mathbf{i}_s = \underbrace{a_{k\ell m}\,b_{\ell mrs}}_{d_{krs}}\,\mathbf{i}_k \circ \mathbf{i}_r \circ \mathbf{i}_s = d_{krs}\,\mathbf{i}_k \circ \mathbf{i}_r \circ \mathbf{i}_s\,. \tag{1.154a} $$

Or simply

$$ d_{krs} = a_{k\ell m}\,b_{\ell mrs}\,. \tag{1.154b} $$

It is obvious that the dot product is a simple, while the double dot product is a double contraction.

EXERCISE 1.9: Rewrite equation (1.31) in indicial notation.
If we take into account that the transpose of $Q_{k'm}$ is $Q_{m\ell'}$ – the primed subscript should be the second – we can write

$$Q_{k'm}Q_{m\ell'} = \delta_{k\ell}\,. \tag{1.155}$$

1.4.6.3. *Special tensors of order four.* We define the unit tensor of order four by the following equation:

$$\boxed{{}^{(4)}\boldsymbol{1} = \delta_{sqkm}\,\mathbf{i}_s \circ \mathbf{i}_q \circ \mathbf{i}_k \circ \mathbf{i}_m\,, \qquad \delta_{sqkm} = \delta_{sk}\delta_{qm}\,.} \tag{1.156}$$

Let $\boldsymbol{W}$ be a tensor of order two. It is not too difficult to check that

$$\boxed{\begin{array}{ll} \delta_{sqkm}\,w_{km} = \delta_{sk}\delta_{qm}w_{km} = w_{sq}\,, & w_{sq}\,\delta_{sqkm} = w_{sq}\,\delta_{sk}\delta_{qm} = w_{km}\,, \\ {}^{(4)}\boldsymbol{1}\cdot\cdot\,\boldsymbol{W} = \boldsymbol{W}\,, & \boldsymbol{W}\cdot\cdot{}^{(4)}\boldsymbol{1} = \boldsymbol{W}\,. \end{array}} \tag{1.157}$$

This result shows that the unit tensor ${}^{(4)}\boldsymbol{1}$ maps a tensor of order two onto itself. The tensor

$$\boxed{{}^{(4)}\boldsymbol{\mathcal{T}} = \mathcal{T}_{sqkm}\mathbf{i}_s \circ \mathbf{i}_q \circ \mathbf{i}_k \circ \mathbf{i}_m, \qquad \mathcal{T}_{sqkm} = \delta_{sm}\delta_{qk}} \tag{1.158}$$

is called transporter since it maps a tensor of order two into its transpose: for the tensor W we get:

$$\boxed{\begin{array}{ll} \mathcal{T}_{sqkm}w_{km} = \delta_{sm}\delta_{qk}w_{km} = w_{qs}, & w_{sq}\mathcal{T}_{sqkm} = w_{sq}\delta_{sm}\delta_{qk} = w_{mk}\,, \\ {}^{(4)}\boldsymbol{\mathcal{T}}\cdot\cdot\boldsymbol{W} = \boldsymbol{W}^T\,, & \boldsymbol{W}\cdot\cdot{}^{(4)}\boldsymbol{\mathcal{T}} = \boldsymbol{W}^T\,. \end{array}} \tag{1.159}$$

The tensor

$$\boxed{{}^{(4)}\boldsymbol{\mathcal{I}} = \mathcal{I}_{sqkm}\mathbf{i}_s \circ \mathbf{i}_q \circ \mathbf{i}_k \circ \mathbf{i}_m, \qquad \mathcal{I}_{sqkm} = \delta_{sq}\delta_{km}} \tag{1.160}$$

maps a tensor of order two into the product of the unit tennsor and its first scalar invariant:

$$\boxed{\begin{array}{ll} \mathcal{I}_{sqkm}w_{km} = \delta_{sq}\delta_{km}w_{km} = \delta_{sq}w_{kk}, & w_{sq}\mathcal{T}_{sqkm} = w_{sq}\delta_{sq}\delta_{km} = \delta_{km}w_{ss}\,, \\ {}^{(4)}\boldsymbol{\mathcal{I}}\cdot\cdot\boldsymbol{W} = \boldsymbol{1}W_I\,, & \boldsymbol{W}\cdot\cdot{}^{(4)}\boldsymbol{\mathcal{I}} = \boldsymbol{1}W_I\,. \end{array}} \tag{1.161}$$

1.4.6.4. *Inverse of a tensor of order four.* Let $\mathcal{C}_{mnk\ell}$ and $\mathcal{S}_{pquv}$ be two tensors of order four. We shall call $\mathcal{S}_{pquv}$ the inverse of $\mathcal{C}_{mnk\ell}$ ($\mathcal{C}_{mnk\ell}$ the inverse of $\mathcal{S}_{pquv}$) if the following relations hold:

$$\boxed{\begin{array}{c} \mathcal{C}_{mnk\ell}\,\mathcal{S}_{k\ell uv} = \delta_{mnuv} = \delta_{mu}\delta_{nv}\,, \\ \mathcal{S}_{pqmn}\,\mathcal{C}_{mnk\ell} = \delta_{pqk\ell} = \delta_{pk}\delta_{q\ell}\,. \end{array}} \tag{1.162}$$

1.4.7. Isotropic tensors. A tensor of order n is said to be isotropic if its components in the basis $\mathbf{i}_\ell$ are the same as those in the basis $\mathbf{i}'_\ell$. If, say, the tensor of order three ${}^{(3)}\boldsymbol{T}$ is isotropic it should hold that

$$t'_{k\ell m} = t_{k\ell m}\,. \tag{1.163}$$

(a) A scalar is obviously isotropic.

(b) The only isotropic vector is the zero vector – the components of a vector change if we rotate the coordinate system about the origin.

(c) All isotropic tensors of order two are of the form $\alpha\delta_{k\ell}$ where α is an arbitrary scalar. The following manipulation shows that a tensor of the form $\alpha\delta_{k\ell}$ is isotropic:

$$\alpha\,\delta'_{k\ell} = \alpha\,\mathbf{i}'_k\cdot\mathbf{i}'_\ell = \underset{(1.26c)}{\uparrow} = \alpha\,Q_{k'm}Q_{\ell' n}\,\mathbf{i}_m\cdot\mathbf{i}_n = \\ = \alpha\,Q_{k'm}Q_{\ell' n}\,\delta_{mn} = \alpha\,Q_{k'm}Q_{m\ell'} = \underset{(1.155)}{\uparrow} = \alpha\,\delta_{k\ell}\,. \tag{1.164}$$

(d) All isotropic tensors of order three (isotropic triads) are of the form $\alpha e_{k\ell r}$. The following manipulation shows that a tensor of the form $\alpha e_{k\ell r}$ is isotropic:

$$\alpha e'_{ijk} = \underset{(B.1.5)}{\uparrow} = \alpha\,Q_{i'p}Q_{j'q}Q_{k'r}\,e_{pqr} = \underset{(1.47)}{\uparrow} = \alpha\,e_{ijk}\det\left(Q_{mn}\right) = \alpha e_{ijk}\,. \tag{1.165}$$

(e) All isotropic tensors of order four (isotropic tetrads) are of the form

$$\boxed{\mathcal{C}_{mnk\ell} = \lambda\delta_{mn}\delta_{k\ell} + \mu\delta_{mk}\delta_{n\ell} + \kappa\delta_{m\ell}\delta_{nk}\,,} \tag{1.166}$$

where λ, μ and κ are scalars. One can check with ease by utilizing the transformation (1.164) that the tensor $C_{mnk\ell}$ is isotropic.

REMARK 1.12: Assume that $\mu = \kappa$. Then the tensor

$$\mathcal{C}_{mnk\ell} = \lambda\delta_{mn}\delta_{k\ell} + \mu\left(\delta_{mk}\delta_{n\ell} + \delta_{m\ell}\delta_{nk}\right)\,. \tag{1.167a}$$

is obviously isotropic and symmetric with respect to the index pairs $_{mn}$ and $_{k\ell}$. It can be checked with ease that its inverse is given by

$$\mathcal{S}_{pqmn} = \frac{1}{4\mu}\left(\delta_{pm}\delta_{qn} + \delta_{pn}\delta_{qm}\right) - \frac{\lambda}{2\mu\left(3\lambda + 2\mu\right)}\delta_{pq}\delta_{mn}\,. \tag{1.167b}$$

Introduce a new constant denoted by ν and assume that

$$\lambda = 2\mu\nu/(1-2\nu)\,. \tag{1.168a}$$

Then

$$\mathcal{C}_{mnk\ell} = \mu\left(\delta_{mk}\delta_{n\ell} + \delta_{m\ell}\delta_{nk}\right) + \frac{2\mu\nu}{1-2\nu}\delta_{mn}\delta_{k\ell} \tag{1.168b}$$

and

$$\mathcal{S}_{pqmn} = \frac{1}{4\mu}\left(\delta_{pm}\delta_{qn} + \delta_{pn}\delta_{qm}\right) - \frac{\nu}{2\mu(1+\nu)}\delta_{pq}\delta_{mn}\,. \tag{1.168c}$$

Assume that ε_{kl} and σ_{mn} are symmetric tensors in the products $\mathcal{C}_{mnk\ell}\varepsilon_{k\ell}$ and $\mathcal{S}_{pqmn}\sigma_{mn}$. For these products we may rewrite both (1.168b) and (1.168c) if we utilize the definition (1.156) of the forth order unit tensor and the definition (1.160) of $\mathcal{I}$. We get

$$\mathcal{C}_{mnk\ell} = 2\mu\delta_{mnkl} + \lambda\mathcal{I}_{mnk\ell} = 2\mu\left(\delta_{mnkl} + \frac{\nu}{1-2\nu}\mathcal{I}_{mnk\ell}\right), \tag{1.169a}$$

$$\mathcal{S}_{pqmn} = \frac{1}{2\mu}\left(\delta_{pqmn} - \frac{\lambda}{3\lambda+2\mu}\mathcal{I}_{pqmn}\right) = \frac{1}{2\mu}\left(\delta_{pqmn} - \frac{\nu}{1+\nu}\mathcal{I}_{pqmn}\right). \tag{1.169b}$$

REMARK 1.13: We have shown that tensors (1.164), (1.165) and (1.166) are all isotropic, we have not proved, however, that tensors other than $\alpha\,\delta_{k\ell}$, αe_{ijk} and the tensor given by equation (1.166) can be isotropic, i.e., the uniqueness has not been proved. In this respect we refer the reader to book [75] and paper [41] on isotropic tensor functions by Richter[6].

1.5. Some elements of tensor analysis

1.5.1. Gradient, curl and divergence. The nabla operator is defined by the following equation

$$\nabla = \underbrace{\frac{\partial}{\partial x_1}}_{\nabla_1}\mathbf{i}_1 + \underbrace{\frac{\partial}{\partial x_2}}_{\nabla_2}\mathbf{i}_2 + \underbrace{\frac{\partial}{\partial x_3}}_{\nabla_3}\mathbf{i}_3 = \frac{\partial}{\partial x_\ell}\mathbf{i}_\ell = \nabla_\ell\mathbf{i}_\ell\,. \tag{1.170a}$$

We shall also apply the following notation convention

$$\frac{\partial}{\partial x_\ell}(\ldots) = (\ldots)_{,\ell}\,. \tag{1.170b}$$

In words: a subscript (here ℓ) preceded by a comma means derivation with respect to the coordinate which belongs to the subscript (with respect to x_ℓ here). Hence, a subscript (here ℓ) in the denominator will be regarded as if it were a subscript in the numerator.

Let $\phi(x_1, x_2, x_3)$ be a scalar field defined in a region of the three dimensional space. If ϕ depends not only on the location but on time as well we write $\phi(x_1, x_2, x_3; t)$. The gradient of the scalar field ϕ is defined by the following equation:

$$\phi\nabla_\ell\mathbf{i}_\ell = \phi_{,\ell}\mathbf{i}_\ell \qquad \text{or} \qquad \phi\nabla_\ell = \phi_{,\ell}\,. \tag{1.171}$$

Let $\boldsymbol{\chi}(x_1, x_2, x_3; t)$ be a vector field. The right gradient (or simply the gradient) of the vector field $\boldsymbol{\chi}$ is defined by the following tensor product:

$$\boldsymbol{\chi}\circ\nabla = (\chi_k\mathbf{i}_k)\circ(\nabla_\ell\mathbf{i}_\ell) = (\chi_k\nabla_\ell)\,\mathbf{i}_k\circ\mathbf{i}_\ell = \chi_{k,\ell}\,\mathbf{i}_k\circ\mathbf{i}_\ell \quad \text{or} \quad \chi_k\nabla_\ell = \chi_{k,\ell}\,. \tag{1.172a}$$

Here

$$[\chi_{k,\ell}] = \begin{bmatrix} \frac{\partial\chi_1}{\partial x_1} & \frac{\partial\chi_1}{\partial x_2} & \frac{\partial\chi_1}{\partial x_3} \\ \frac{\partial\chi_2}{\partial x_1} & \frac{\partial\chi_2}{\partial x_2} & \frac{\partial\chi_2}{\partial x_3} \\ \frac{\partial\chi_3}{\partial x_1} & \frac{\partial\chi_3}{\partial x_2} & \frac{\partial\chi_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \chi_{1,1} & \chi_{1,2} & \chi_{1,3} \\ \chi_{2,1} & \chi_{2,2} & \chi_{2,3} \\ \chi_{3,1} & \chi_{3,2} & \chi_{3,3} \end{bmatrix}. \tag{1.172b}$$

[6] Hans Richter (1912-1978)

The gradient of the vector field $\boldsymbol{\chi}$ is a tensor field.

The divergence of the vector field $\boldsymbol{\chi}$ is the trace of its gradient:

$$\boldsymbol{\chi}\cdot\nabla = (\chi_k \mathbf{i}_k)\cdot(\nabla_\ell \mathbf{i}_\ell) = (\chi_k \nabla_\ell)\,\mathbf{i}_k\cdot\mathbf{i}_\ell = (\chi_k\nabla_\ell)\,\delta_{k\ell} = \\ = \chi_\ell\nabla_\ell = \chi_{\ell,\ell} = \chi_{1,1}+\chi_{2,2}+\chi_{3,3}\,. \quad (1.173)$$

The curl of the vector field $\boldsymbol{\chi}$ is defined by the following cross product:

$$\text{curl}\,\boldsymbol{\chi} = \nabla\times\boldsymbol{\chi} = \nabla_k \mathbf{i}_k \times \chi_\ell \mathbf{i}_\ell = \nabla_k\chi_\ell\,\mathbf{i}_k\times\mathbf{i}_\ell = \underset{(1.43)}{\uparrow} = e_{k\ell r}\chi_{\ell,k}\mathbf{i}_r\,, \\ \text{or}\quad e_{k\ell r}\nabla_k\chi_\ell = e_{k\ell r}\chi_{\ell,k}\,. \quad (1.174)$$

The vector field $\boldsymbol{\chi}$ is said to be conservative or rotation free if curl $\boldsymbol{\chi} = \mathbf{0}$.

Assume that there exists a scalar function $\phi(x_1,x_2,x_3)$ such that $\boldsymbol{\chi} = \nabla\phi(x_1,x_2,x_3)$. Then

$$\text{curl}\,\boldsymbol{\chi} = \nabla\times\nabla\phi = \mathbf{0}\,, \quad (1.175)$$

which means that the vector field $\boldsymbol{\chi}$ is rotation free. The function $\phi(x_1,x_2,x_3)$ is called potential function.

Let $\boldsymbol{T} = t_{k\ell}\,\mathbf{i}_k\circ\mathbf{i}_\ell$, $t_{k\ell} = t_{k\ell}\,(x_1,x_2,x_3;t)$ be a tensor field. Its gradient is defined by the following relationship:

$$\boldsymbol{T}\circ\nabla = (t_{k\ell}\,\mathbf{i}_k\circ\mathbf{i}_\ell)\circ(\mathbf{i}_r\nabla_r) = t_{k\ell}\nabla_r\,\mathbf{i}_k\circ\mathbf{i}_\ell\circ\mathbf{i}_r\,,\quad \text{or}\quad t_{k\ell}\nabla_r = t_{k\ell,r}\,. \quad (1.176)$$

The gradient of $\boldsymbol{T}$ is a tensor of order three (a triad).

The divergence of rhe tensor field $\boldsymbol{T}$ is defined by the following dot product:

$$\boldsymbol{T}\cdot\nabla = (t_{k\ell}\,\mathbf{i}_k\circ\mathbf{i}_\ell)\cdot(\mathbf{i}_r\nabla_r) = t_{k\ell}\nabla_r\,\mathbf{i}_k\,\underbrace{\mathbf{i}_\ell\cdot\mathbf{i}_r}_{\delta_{\ell r}} = \mathbf{i}_k\,t_{k\ell}\nabla_\ell\,, \quad (1.177a)$$

or simply

$$\boxed{t_{k\ell}\nabla_\ell = t_{k\ell,\ell}\,.} \quad (1.177b)$$

The divergence of $\boldsymbol{T}$ is a vector field.

The Laplace operator (or Laplacian) is defined by the following dot product:

$$\boxed{\Delta = \nabla\cdot\nabla = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\,.} \quad (1.178)$$

1.5.2. Integral theorems.

1.5.2.1. *Divergence theorem.* Let A be a finite closed surface enclosing a volume V. Further let $\boldsymbol{H} = H_{k\ell}\,\mathbf{i}_k\circ\mathbf{i}_\ell$ be a continuously differentiable tensor field defined on V. The outward unit normal to A is denoted by $\mathbf{n}$. Equation

$$\boxed{\begin{gathered}\int_V \boldsymbol{H}\cdot\nabla\,\mathrm{d}V = \int_A \boldsymbol{H}\cdot\mathbf{n}\,\mathrm{d}A\,,\\ \int_V H_{k\ell}\nabla_\ell\,\mathrm{d}V = \int_V H_{k\ell,\ell}\,\mathrm{d}V = \int_A H_{k\ell}n_\ell\,\mathrm{d}A\end{gathered}} \quad (1.179)$$

is the divergence (or Gauss) theorem[7]. With the aid of the Gauss theorem one can transform a volume integral into a surface integral and vice versa.

Let $\boldsymbol{t}$ be a sufficiently smooth tensor field on V. Further let $\mathbf{u}$ be a sufficiently smooth vector field regarded also on V. Consider now the volume integral

$$\int_V \mathbf{u}\cdot(\overset{\downarrow}{\boldsymbol{t}}\cdot\nabla)\,\mathrm{d}V$$

in which a down arrow shows the quantity to which the operator ∇ is applied. By utilizing (a) the product rule of derivation, (b) the divergence theorem (1.179) and then (c) property $(1.96)_3$ of the inner product we can manipulate the above integral into the following form

$$\begin{aligned}
\int_V \mathbf{u}\cdot(\overset{\downarrow}{\boldsymbol{t}}\cdot\nabla)\mathrm{d}V &= \int_V \left[(\mathbf{u}\cdot\boldsymbol{t})\cdot\nabla - \overset{\downarrow}{\mathbf{u}}\cdot\boldsymbol{t}\cdot\nabla\right]\mathrm{d}V = \\
&= \int_A \mathbf{u}\cdot\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A - \int_V \boldsymbol{t}\cdot\cdot\,(\mathbf{u}\circ\nabla)\,\mathrm{d}V\,, \\
\int_V u_k t_{k\ell,\ell}\mathrm{d}V &= \int_V [\underbrace{u_k t_{k\ell,\ell} + u_{k,\ell}t_{k\ell}}_{(u_k t_{k\ell})_{,\ell}} - u_{k,\ell}t_{k\ell}]\mathrm{d}V = \\
&= \int_V \left[(u_k t_{k\ell})_{,\ell} - u_{k,\ell}t_{k\ell}\right]\mathrm{d}V = \int_A u_k t_{k\ell} n_\ell\,\mathrm{d}A - \int_V t_{k\ell}\,u_{k,\ell}\,\mathrm{d}V\,.
\end{aligned} \tag{1.180}$$

This equation is the rule of partial integration.

1.5.2.2. *The Stokes theorem.* Now let S be an open surface. The positive description on the closed curve g bounding S (the positive direction for the arc coordinate s) is the one which leaves the surface on the left – see Figure 1.12.

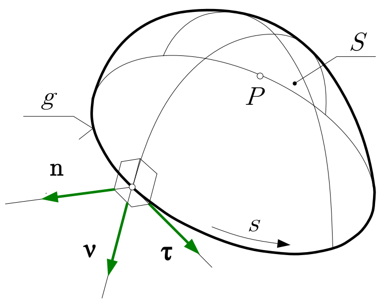

FIGURE 1.12. An open surface

The unit normal on S is $\mathbf{n}$, the unit tangent to g is denoted by $\boldsymbol{\tau}$, the vector $\boldsymbol{\nu} = \boldsymbol{\tau}\times\boldsymbol{n}$ is the binormal. Let $\boldsymbol{H}$ be a differentiable tensor field both in A and

[7]Carl Friderich Gauss (1777-1855)

in the neighborhood of S. The Stokes[8] theorem reads as follows

$$\boxed{\begin{aligned}\int_S \overset{\downarrow}{\boldsymbol{H}} \cdot (\mathbf{n} \times \nabla)\, \mathrm{d}S &= \oint_g \boldsymbol{H} \cdot \boldsymbol{\tau}\, \mathrm{d}s\,, \\ \int_S (H_{kp}\nabla_r)\, n_\ell e_{k\ell r}\, \mathrm{d}S &= \oint_g H_{kp}\tau_p\, \mathrm{d}s\,.\end{aligned}} \tag{1.181}$$

1.5.2.3. *Scalar valued tensor functions.* The scalar field $f(\boldsymbol{E}) = f(E_{k\ell})$ in which the tensor field $E_{k\ell}$ is the independent variable is referred to as scalar valued tensor function or real tensor function. Its increment is given by

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial E_{k\ell}}\,\mathrm{d}E_{k\ell} = \frac{\partial f}{\partial E_{k\ell}}\,\underbrace{\delta_{km}\delta_{\ell n}\mathrm{d}E_{mn}}_{\mathrm{d}E_{k\ell}} = \frac{\partial f}{\partial E_{k\ell}}\,\underbrace{\mathbf{i}_k \cdot \mathbf{i}_m}_{\delta_{km}}\;\underbrace{\mathbf{i}_\ell \cdot \mathbf{i}_n}_{\delta_{\ell n}}\,\mathrm{d}E_{mn} = \\ &\underset{(\mathbf{i}_k\cdot\mathbf{i}_m)(\mathbf{i}_\ell\cdot\mathbf{i}_n)=(\mathbf{i}_k\circ\mathbf{i}_\ell)\cdot\cdot(\mathbf{i}_m\circ\mathbf{i}_n)}{=\uparrow} \left(\frac{\partial f}{\partial E_{k\ell}}\,\mathbf{i}_k \circ \mathbf{i}_\ell\right) \cdot\cdot\, (\mathrm{d}E_{mn}\,\mathbf{i}_m \circ \mathbf{i}_n)\,,\end{aligned} \tag{1.182}$$

where the tensor of order two

$$\frac{\partial f}{\partial \boldsymbol{E}} = \frac{\partial f}{\partial E_{k\ell}}\,\mathbf{i}_k \circ \mathbf{i}_\ell \tag{1.183}$$

is the derivative of f. In applications the tensor $\boldsymbol{E}$ is in general a symmetric tensor.

1.6. Curvilinear coordinate systems

1.6.1. Important properties of curvilinear coordinate systems. In a curvilinear coordinate system the coordinate lines, along which only one coordinate changes and the other two are constants, are in general not straight lines but space curves. The coordinate surfaces, on which one coordinate is constant and the other two changes, are not planes but curved surfaces. The tangents (unit tangents) of the coordinate lines form a vector triad or a basis since any vector or tensor field can be given in terms of these vectors. The main problem is that this basis is not constant but changes in the 3D space.

1.6.2. Cylindrical coordinate system.
1.6.2.1. Figure 1.13. shows a cylindrical coordinate system. Figure 1.13. also represents the coinciding coordinate lines of the two Cartesian coordinate systems (xyz) and $(x_1x_2x_3)$ we have applied so far. As regards the point P in the cylindrical coordinate system (R, ϑ, z) (a) the coordinate R is the perpendicular

[8] George Stokes (1819-1903), Lord Baron Kelvin (1824-1907); the first and fundamental form of the theorem is in letter [72] sent to Stokes in 1850 by Kelvin.

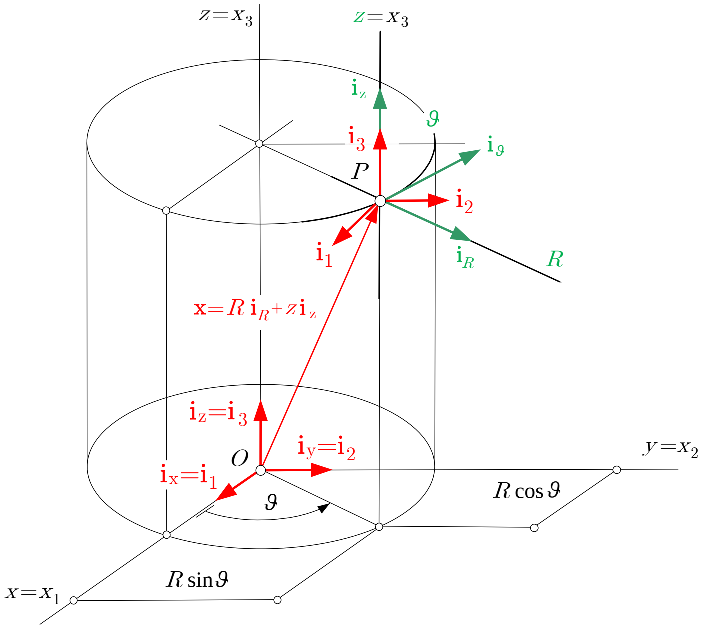

FIGURE 1.13. Cylindrical coordinate system

distance of P from the coordinate axis $z = x_3$, (b) the coordinate ϑ ($\vartheta \in [-\pi, \pi]$, ϑ is positive in Figure 1.13) is the angle formed by the perpendicular projection of the position vector $\mathbf{x}$ on the coordinate plane $(xy) = (x_1x_2)$ and the coordinate axis $x = x_1$ (often called polar angle) and (c) the coordinate z is the same as the coordinate x_3 in the coordinate system (xyz), i.e., it is the perpendicular projection of the point P on the coordinate axis x_3. The coordinate surface $R =$ constant is a cylinder of radius R with a centroidal axis coinciding with the coordinate axis z. It is obvious that the coordinate ϑ is undetermined if the point P is on the axis z.

The coordinate triplets R, ϑ, z and x_1, x_2, x_3 are related to each other via the following relations:

$$x_1 = R\cos\vartheta\,, \qquad x_2 = R\sin\vartheta\,, \qquad x_3 = z \tag{1.184a}$$

or

$$R = \sqrt{x_1^2 + x_2^2}\,, \qquad \vartheta = \tan^{-1}\frac{x_2}{x_1}\,, \qquad z = x_3\,. \tag{1.184b}$$

The vector $\mathbf{i}_R$ is the unit tangent to the coordinate line $\vartheta =$ constant, $z =$ constant. Similarly the vector $\mathbf{i}_\vartheta$ is the unit tangent to the coordinate line $R =$ constant, $z =$ constant. It is clear from Figure 1.13 that

$$\mathbf{x} = R\cos\vartheta\,\mathbf{i}_1 + R\sin\vartheta\,\mathbf{i}_2 + z\,\mathbf{i}_3 \tag{1.185}$$

is the position vector. Hence,

$$\mathbf{i}_R = \frac{\partial \mathbf{x}}{\partial R} = \cos\vartheta\,\mathbf{i}_1 + \sin\vartheta\,\mathbf{i}_2\,, \quad \mathbf{i}_\vartheta = \frac{\partial \mathbf{x}}{R\partial\vartheta} = -\sin\vartheta\,\mathbf{i}_1 + \cos\vartheta\,\mathbf{i}_2\,, \\ \mathbf{i}_z = \frac{\partial \mathbf{x}}{\partial z} = \mathbf{i}_3 \tag{1.186a}$$

are the unit tangents to the coordinate lines at the point P. It can now be seen with ease that

$$\boxed{\frac{\mathrm{d}\mathbf{i}_R}{\mathrm{d}\vartheta} = \mathbf{i}_\vartheta \qquad \text{and} \qquad \frac{\mathrm{d}\mathbf{i}_\vartheta}{\mathrm{d}\vartheta} = -\mathbf{i}_R\,.} \tag{1.186b}$$

Given the vectors $\mathbf{i}_R$, $\mathbf{i}_\vartheta$ and $\mathbf{i}_z$ equations (1.186a) can be solved for $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$:

$$\mathbf{i}_1 = \cos\vartheta\,\mathbf{i}_R - \sin\vartheta\,\mathbf{i}_\vartheta\,, \qquad \mathbf{i}_2 = \sin\vartheta\,\mathbf{i}_R + \cos\vartheta\,\mathbf{i}_\vartheta\,, \qquad \mathbf{i}_3 = \mathbf{i}_z\,. \tag{1.186c}$$

Remark 1.14: Any vector or tensor field which describes the mechanical behavior of a material point in an elastic body is regarded in a local basis attached to the material point itself. If we work in a Cartesian coordinate system the local basis is the same at every point within the body and can also be referred to as a global basis which is usually attached mentally to the origin O. Figure 1.13 shows these bases (formed by the orthonormal vector triad $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$) at P and O as well.
The unit vectors $\mathbf{i}_R$, $\mathbf{i}_\vartheta$, $\mathbf{i}_z$ given by equation (1.186a) meet the orthogonality condition

$$\mathbf{i}_m \cdot \mathbf{i}_n = 0\,, \qquad (m, n = R, \vartheta, z;\; m \neq n)\,, \tag{1.187}$$

therefore, they form an orthonormal vector triad and constitute the local basis in cylindrical coordinate systems. It is worth emphasizing that the base vectors $\mathbf{i}_R$, $\mathbf{i}_\vartheta$ are not constants and this fact should be taken into account when the derivatives of vector and tensor fields are to be determined in the cylindrical coordinate system $(R\vartheta z)$. Figure 1.13 also shows the local basis constituted by the vector triad $\mathbf{i}_R$, $\mathbf{i}_\vartheta$ and $\mathbf{i}_z$ at the point P.

1.6.2.2. Let $\mathbf{u}(R, \vartheta, z)$ be a vector field given in the coordinate system $(R\vartheta z)$. We can give it in the form

$$\mathbf{u} = u_R(R, \vartheta, z)\,\mathbf{i}_R + u_\vartheta(R, \vartheta, z)\,\mathbf{i}_\vartheta + u_z(R, \vartheta, z)\,\mathbf{i}_z\,, \tag{1.188}$$

where u_R, u_ϑ and u_z are the (scalar) components of the vector $\mathbf{u}$.

It is obvious that

$$\mathbf{x} = R\,\mathbf{i}_R + z\,\mathbf{i}_z \tag{1.189}$$

is the position vector in the local basis.

Further let $\boldsymbol{t}(R, \vartheta, z)$ be a tensor field considered again in the coordinate system $(R\vartheta z)$. If we know the image vectors

$$\begin{aligned} \mathbf{t}_R &= t_{RR}\,\mathbf{i}_R + t_{\vartheta R}\,\mathbf{i}_\vartheta + t_{zR}\,\mathbf{i}_z\,, \\ \mathbf{t}_\vartheta &= t_{R\vartheta}\,\mathbf{i}_R + t_{\vartheta\vartheta}\,\mathbf{i}_\vartheta + t_{z\vartheta}\,\mathbf{i}_z\,, \\ \mathbf{t}_z &= t_{Rz}\,\mathbf{i}_R + t_{\vartheta z}\,\mathbf{i}_\vartheta + t_{zz}\,\mathbf{i}_z \end{aligned} \tag{1.190}$$

which belong to the local base vectors $\mathbf{i}_R$, $\mathbf{i}_\vartheta$, $\mathbf{i}_z$ we obtain the tensor in the form

$$\boldsymbol{t} = \mathbf{t}_R \circ \mathbf{i}_R + \mathbf{t}_\vartheta \circ \mathbf{i}_\vartheta + \mathbf{t}_z \circ \mathbf{i}_z \,. \tag{1.191}$$

Its matrix can be given in terms of the column matrices $\underset{(3\times1)}{\underline{\mathbf{t}_R}}$, $\underset{(3\times1)}{\underline{\mathbf{t}_\vartheta}}$, $\underset{(3\times1)}{\underline{\mathbf{t}_z}}$:

$$\underset{(3\times1)}{\underline{\mathbf{t}}} = \left[\, \underset{(3\times1)}{\underline{\mathbf{t}_R}} \;\middle|\; \underset{(3\times1)}{\underline{\mathbf{t}_\vartheta}} \;\middle|\; \underset{(3\times1)}{\underline{\mathbf{t}_z}} \,\right] = \begin{bmatrix} t_{RR} & t_{R\vartheta} & t_{Rz} \\ t_{\vartheta R} & t_{\vartheta\vartheta} & t_{\vartheta z} \\ t_{zR} & t_{z\vartheta} & t_{zz} \end{bmatrix}. \tag{1.192}$$

Here t_{RR}, $t_{\vartheta R}$, t_{zR}, $t_{R\vartheta}, \ldots, t_{zz}$ are the (scalar) components of the tensor $\boldsymbol{t}$ in the coordinate system $(R\vartheta z)$.

Remark 1.15: The Cartesian coordinate system and the cylindrical coordinate system have a common property: the local base vectors constitute an orthonormal triad. Assume that the subscripts R, ϑ, z of the coordinate system $(R\vartheta z)$ correspond to the subscripts 1, 2, 3. It follows from the common property we have mentioned and the assumed correspondences between the subscripts that we can apply all the notational conventions we introduced when dealing with indicial notations and tensor algebra in Sections 1.4 and 1.4.7.

Consider for instance the product $\boldsymbol{T} \cdot \mathbf{n}$ $(|\mathbf{n}| = 1)$ which is the image $\mathbf{t}^{(n)}$ of the unit vector $\mathbf{n}$. In indicial notation we can write

$$t_k^{(n)} = t_{k\ell} n_\ell \tag{1.193a}$$

which yields on the basis of the correspondences $1 \leftrightarrow R$, $2 \leftrightarrow \vartheta$, $3 \leftrightarrow z$ that

$$\begin{aligned} t_R^{(n)} &= t_{RR}\, n_R + t_{R\vartheta} n_\vartheta + t_{Rz} n_z \,, \\ t_\vartheta^{(n)} &= t_{\vartheta R}\, n_R + t_{\vartheta\vartheta} n_\vartheta + t_{\vartheta z} n_z \,, \\ t_z^{(n)} &= t_{zR}\, n_R + t_{z\vartheta} n_\vartheta + t_{zz} n_z \,. \end{aligned} \tag{1.193b}$$

Assume that the cylindrical coordinate system is the primed coordinate system, i.e., $\mathbf{i}'_1 = \mathbf{i}_R$, $\mathbf{i}'_2 = \mathbf{i}_\vartheta$, $\mathbf{i}'_3 = \mathbf{i}_z$. Making use of relations (1.30) we get that

$$\begin{aligned} \underset{(3\times1)}{\underline{\mathbf{u}'}} &= \begin{bmatrix} u_R \\ u_\vartheta \\ u_z \end{bmatrix} = \begin{bmatrix} Q_{1'1} & Q_{1'2} & Q_{1'3} \\ Q_{2'1} & Q_{2'2} & Q_{2'3} \\ Q_{3'1} & Q_{3'2} & Q_{3'3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{i}'_1 \cdot \mathbf{i}_1 & \mathbf{i}'_1 \cdot \mathbf{i}_2 & \mathbf{i}'_1 \cdot \mathbf{i}_3 \\ \mathbf{i}'_2 \cdot \mathbf{i}_1 & \mathbf{i}'_2 \cdot \mathbf{i}_2 & \mathbf{i}'_2 \cdot \mathbf{i}_3 \\ \mathbf{i}'_3 \cdot \mathbf{i}_1 & \mathbf{i}'_3 \cdot \mathbf{i}_2 & \mathbf{i}'_3 \cdot \mathbf{i}_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \underset{(1.186a)}{=\uparrow=} \\ &= \begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underset{(3\times3)}{\underline{\mathbf{Q}}}\ \underset{(3\times1)}{\underline{\mathbf{u}}} \end{aligned} \tag{1.194a}$$

or

$$u'_k = Q_{k'\ell} u_\ell \,, \tag{1.194b}$$

where $[Q_{k'\ell}]$ is the transformation matrix between the coordinate systems $(R\vartheta z)$ and $(x_1 x_2 x_3)$.

1.6.2.3. The nabla operator in the cylindrical coordinate system $(R\vartheta z)$ is defined by the following relation:

$$\boxed{\nabla = \frac{\partial}{\partial R}\mathbf{i}_R + \frac{1}{R}\frac{\partial}{\partial \vartheta}\mathbf{i}_\vartheta + \frac{\partial}{\partial z}\mathbf{i}_z\,.} \tag{1.195}$$

Let $\phi(R,\vartheta,z)$ be a scalar field. Its gradient is

$$\phi\,\nabla = \frac{\partial \phi}{\partial R}\mathbf{i}_R + \frac{1}{R}\frac{\partial \phi}{\partial \vartheta}\mathbf{i}_\vartheta + \frac{\partial \phi}{\partial z}\mathbf{i}_z\,. \tag{1.196}$$

Consider now the vector field

$$\mathbf{v}(R,\vartheta,z) = v_R(R,\vartheta,z)\,\mathbf{i}_R(\vartheta) + v_\vartheta(R,\vartheta,z)\,\mathbf{i}_\vartheta(\vartheta) + v_z(R,\vartheta,z)\,\mathbf{i}_z\,. \tag{1.197}$$

The right gradient of the vector field $\mathbf{v}$ in the cylindrical coordinate system and its matrix are given by the following relations:

$$\begin{aligned}
\boldsymbol{l} = \mathbf{v}\circ\nabla &\underset{(1.195)}{\uparrow}= \frac{\partial \mathbf{v}}{\partial R}\circ\mathbf{i}_R + \frac{1}{R}\frac{\partial \mathbf{v}}{\partial \vartheta}\circ\mathbf{i}_\vartheta + \frac{\partial \mathbf{v}}{\partial z}\circ\mathbf{i}_z \underset{(1.186\text{b})}{\uparrow}= \\
&= \left(\frac{\partial v_R}{\partial R}\,\mathbf{i}_R + \frac{\partial v_\vartheta}{\partial R}\,\mathbf{i}_\vartheta + \frac{\partial v_z}{\partial R}\,\mathbf{i}_R\right)\circ\mathbf{i}_R + \\
&+ \left[\left(\frac{1}{R}\frac{\partial v_R}{\partial \vartheta} - \frac{v_\vartheta}{R}\right)\mathbf{i}_R + \left(\frac{1}{R}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{v_R}{R}\,\mathbf{i}_\vartheta\right) + \frac{\partial v_z}{\partial \vartheta}\,\mathbf{i}_R\right]\circ\mathbf{i}_\vartheta + \\
&+ \left(\frac{\partial v_R}{\partial z}\,\mathbf{i}_R + \frac{\partial v_\vartheta}{\partial z}\,\mathbf{i}_\vartheta + \frac{\partial v_z}{\partial z}\,\mathbf{i}_R\right)\circ\mathbf{i}_z = \\
&= \boldsymbol{l}_R\circ\mathbf{i}_R + \boldsymbol{l}_\vartheta\circ\mathbf{i}_\vartheta + \boldsymbol{l}_z\circ\mathbf{i}_z\,,
\end{aligned} \tag{1.198}$$

$$\underset{(3\times3)}{\underline{\boldsymbol{l}}} = \left[\ \underset{(3\times1)}{\underline{\boldsymbol{l}_R}}\ \middle|\ \underset{(3\times1)}{\underline{\boldsymbol{l}_\vartheta}}\ \middle|\ \underset{(3\times1)}{\underline{\boldsymbol{l}_z}}\ \right] = \begin{bmatrix}
\dfrac{\partial v_R}{\partial R} & \dfrac{1}{R}\dfrac{\partial v_R}{\partial \vartheta} - \dfrac{v_\vartheta}{R} & \dfrac{\partial v_R}{\partial z} \\
\dfrac{\partial v_\vartheta}{\partial R} & \dfrac{1}{R}\dfrac{\partial v_\vartheta}{\partial \vartheta} + \dfrac{v_R}{R} & \dfrac{\partial v_\vartheta}{\partial z} \\
\dfrac{\partial v_z}{\partial R} & \dfrac{\partial v_z}{\partial \vartheta} & \dfrac{\partial v_z}{\partial z}
\end{bmatrix}. \tag{1.199}$$

Tensor $\boldsymbol{l}$ can be resolved into symmetric and skew parts:

$$\boldsymbol{l} = \underbrace{\frac{1}{2}\left(\boldsymbol{l} + \boldsymbol{l}^T\right)}_{\boldsymbol{d}} + \underbrace{\frac{1}{2}\left(\boldsymbol{l} - \boldsymbol{l}^T\right)}_{\boldsymbol{\Omega}} = \boldsymbol{d} + \boldsymbol{\Omega}\,. \tag{1.200}$$

The divergence of the vector field $\mathbf{v}$ can be obtained from (1.198):

$$\mathbf{v}\cdot\nabla = \frac{\partial v_R}{\partial R} + \frac{1}{R}\frac{\partial v_\vartheta}{\partial \vartheta} + \frac{v_R}{R} + \frac{\partial v_R}{\partial R} = d_I\,, \tag{1.201}$$

where d_I is the first scalar invariant of the tensor $\boldsymbol{d}$. The divergence of the gradient $\phi\circ\nabla$ can be calculated similarly:

$$(\phi\circ\nabla)\cdot\nabla = \left(\frac{\partial \phi}{\partial R}\,\mathbf{i}_R + \frac{1}{R}\frac{\partial \phi}{\partial \vartheta}\,\mathbf{i}_\vartheta + \frac{\partial \phi}{\partial z}\,\mathbf{i}_z\right)\cdot\left(\frac{\partial}{\partial R}\,\mathbf{i}_R + \frac{1}{R}\frac{\partial}{\partial \vartheta}\,\mathbf{i}_\vartheta + \frac{\partial}{\partial z}\,\mathbf{i}_z\right) =$$

$$= \underset{(1.186b)}{\uparrow} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \phi}{\partial \vartheta^2} + \frac{\partial^2 \phi}{\partial z^2} = \Delta\phi \quad (1.202)$$

Here

$$\boxed{\Delta = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial z^2}} \quad (1.203)$$

is the Laplace operator (Laplacian) in the cylindrical coordinate system. If we apply it to the vector field

$$\mathbf{u}(R,\vartheta,z) = u_R(R,\vartheta,z)\,\mathbf{i}_R(\vartheta) + u_\vartheta(R,\vartheta,z)\,\mathbf{i}_\vartheta(\vartheta) + u_z(R,\vartheta,z)\,\mathbf{i}_z \quad (1.204)$$

we get

$$\Delta\mathbf{u} = \left(\Delta u_R - \frac{2}{R^2}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_R}{R^2}\right)\mathbf{i}_R + \left(\Delta u_\vartheta + \frac{2}{R^2}\frac{\partial u_R}{\partial \vartheta} - \frac{u_\vartheta}{R^2}\right)\mathbf{i}_\vartheta + \Delta u_z\,\mathbf{i}_z\,. \quad (1.205)$$

If we take into account that $\mathrm{d}\mathbf{x} = \mathrm{d}R\,\mathbf{i}_R + R\mathrm{d}\vartheta\,\mathbf{i}_\vartheta + \mathrm{d}z\,\mathbf{i}_z$ we can determine the linear part of the change in the vector field $\mathbf{v}$ caused by the change $\mathrm{d}\mathbf{x}$ in the position vector:

$$\mathrm{d}\mathbf{v} = \boldsymbol{l}\cdot\mathrm{d}\mathbf{x} = (\mathbf{v}\circ\nabla)\cdot\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{v}}{\partial R}\mathrm{d}R + \frac{\partial \mathbf{v}}{\partial \vartheta}\mathrm{d}\vartheta + \frac{\partial \mathbf{v}}{\partial z}\mathrm{d}z\,. \quad (1.206)$$

1.7. Problems

PROBLEM 1.1: We know the coordinates of the points A, B and C in the coordinate system $(x_1x_2x_3)$: $A(2;0;5)$ m, $B(-1;4;0)$ m, $C(-3;0;4)$ m.

(a) Determine the angle α at vertex A in the triangle ABC.
(b) Calculate the area of the triangle ABC and the volume of the tetrahedron $OABC$.

PROBLEM 1.2: Assume that the sum of three vectors vanishes: $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$. Prove that

$$\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}\,.$$

PROBLEM 1.3: Prove equations (1.16).

PROBLEM 1.4: Show that the matrix

$$\underset{(3\times3)}{\underline{\mathbf{Q}}} = \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 \\ 1/2 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

is a proper orthogonal matrix.

PROBLEM 1.5: Show that the transformation matrices are popper orthogonal matrices.

PROBLEM 1.6: Give the unabridged form of each equation listed below. If the indicial notation is used incorrectly explain why.

$$(a)\ \ F_i = G_i + H_{ij}\,a_j\,, \qquad (b)\ \ u_i = v_j\,,$$

$$(c)\ \ F_\ell = A_\ell + B_{\ell j}\,C_jD_j\,, \qquad (d)\ \ \Psi_\ell = \frac{\partial \Phi}{\partial x_\ell}\,,$$

$$(e)\ \ d=\sqrt{x_k\,x_k}\,, \qquad\qquad (f)\ \ t_\alpha=\sigma_{\alpha\beta}n_\beta\,.$$

PROBLEM 1.7: Simplify the following expressions:

$$(a)\ \ D_{pqr}\delta_{rs}\,,\quad (b)\ \ F_{k\ell m}\delta_{\ell m}\,,\quad (c)\ \ c_{pqrs}\delta_{pm}\delta_{qn}\,,\quad (d)\ \ a_{k\ell}\delta_{\ell r}\delta_{qs}\,.$$

PROBLEM 1.8: Show that equation (1.43) is equivalent to equations $(1.14)_2$.

PROBLEM 1.9: Show that the expression $|a_{kl}|=e_{pqr}a_{p1}a_{q2}a_{r3}$ is the expansion of the determinant by columns.

PROBLEM 1.10: Let $\mathbf{n}$ ($|\mathbf{n}|=1$) be the normal to the plane S that passes through the origin. Further let $\mathbf{r}_\perp$ be that component of the position vector $\mathbf{r}$ which lies in the plane S. Show that $\mathbf{r}_\perp$ is given by the mapping $\mathbf{r}_\perp=\boldsymbol{W}\cdot\mathbf{r}$ where

$$\boldsymbol{W}=\boldsymbol{1}-\mathbf{n}\circ\mathbf{n}\,. \tag{1.207}$$

PROBLEM 1.11: Let P be the tip of the position vector $\mathbf{r}$ in the previous problem. Show that the reflection P'' of the point P with respect to the plane S is given by the mapping $\mathbf{r}_{O_wP''}=\boldsymbol{W}\cdot\mathbf{r}$ where

$$\boldsymbol{W}=\boldsymbol{1}-2\mathbf{n}\circ\mathbf{n}\,. \tag{1.208}$$

PROBLEM 1.12: What is the matrix of tensor (1.207)?

PROBLEM 1.13: What is the matrix of tensor (1.208)?

PROBLEM 1.14: Prove that

$$w_{mn}=Q_{mk'}Q_{n\ell'}\,w'_{k\ell}\,. \tag{1.209}$$

PROBLEM 1.15: Show that relations (1.83) follow from the definition given for the transpose of a tensor.

PROBLEM 1.16: Show that (a) $s_{k\ell}=s_{\ell k}$ if $\boldsymbol{S}$ is symmetric; (b) $s_{k\ell}=s_{\ell k}$, i.e., $s_{11}=s_{22}=s_{33}=0$, $s_{12}=-s_{21}$, $s_{13}=-s_{31}$ and $s_{23}=-s_{32}$ if $\boldsymbol{S}$ is skew.

PROBLEM 1.17: Assume that we know the axial vector that belongs to the skew tensor $\boldsymbol{S}=\boldsymbol{S}_{\mathsf{skew}}$. Show that the matrix of the tensor can be given in terms of the components of the axial vector in the following form:

$$\underset{(3\times3)}{\underline{\boldsymbol{S}}}=[s_{k\ell}]=\begin{bmatrix}0 & -s_3^{(a)} & s_2^{(a)}\\ s_3^{(a)} & 0 & -s_1^{(a)}\\ -s_2^{(a)} & s_1^{(a)} & 0\end{bmatrix}. \tag{1.210}$$

PROBLEM 1.18: Given the matrix of a tensor $\boldsymbol{T}$ in the coordinate system $(x_1x_2x_3)$:

$$\underset{(3\times3)}{\underline{\mathbf{T}}}=\begin{bmatrix}80 & 0 & 0\\ 0 & 40 & -32\\ 0 & -32 & -80\end{bmatrix}\quad [\mathrm{N/mm^2}]\,. \tag{1.211}$$

Find the tensor components $t'_{k\ell}$ if

$$\mathbf{i}'_1=\mathbf{i}_1\,,\qquad \mathbf{i}'_2=\frac{1}{\sqrt{17}}\left(4\mathbf{i}_2-\mathbf{i}_3\right)\,,\qquad \mathbf{i}'_3=\frac{1}{\sqrt{17}}\left(\mathbf{i}_2+4\mathbf{i}_3\right)\,. \tag{1.212}$$

PROBLEM 1.19: Let $\boldsymbol{S} = s_{pq}\,\mathbf{i}_p \circ \mathbf{i}_q$, $\boldsymbol{T} = t_{rs}\,\mathbf{i}_r \circ \mathbf{i}_s$ and $\boldsymbol{W} = w_{k\ell}\,\mathbf{i}_k \circ \mathbf{i}_\ell$ be three tensors. Prove that

$$\boldsymbol{S} \cdot\cdot\, (\boldsymbol{T} \cdot \boldsymbol{W}) = \left(\boldsymbol{T}^T \cdot \boldsymbol{S}\right) \cdot\cdot\, \boldsymbol{W} = \left(\boldsymbol{S} \cdot \boldsymbol{W}^T\right) \cdot\cdot\, \boldsymbol{T}\,. \tag{1.213}$$

PROBLEM 1.20: Given the tensor $\boldsymbol{W}$ in the form

$$\boldsymbol{W} = \kappa\,(\boldsymbol{1} - \mathbf{i}_1 \circ \mathbf{i}_1) + \gamma\,(\mathbf{i}_1 \circ \mathbf{i}_2 + \mathbf{i}_2 \circ \mathbf{i}_1)\ . \tag{1.214}$$

Show that the eigenvalues and eigenvectors are as follows:

$$\begin{aligned}
\lambda_1 &= \frac{\kappa}{2} - \frac{1}{2}\sqrt{\kappa^2 + 4\gamma^2}\,, & \mathbf{n}_1 &= \frac{1}{\sqrt{1 + \left(\frac{\lambda_3}{\gamma}\right)^2}}\left(-\frac{\lambda_3}{\gamma}\mathbf{i}_1 + \mathbf{i}_2\right);\\
\lambda_2 &= \kappa\,, & \mathbf{n}_2 &= \mathbf{i}_3\,;\\
\lambda_3 &= \frac{\kappa}{2} + \frac{1}{2}\sqrt{\kappa^2 + 4\gamma^2}\,, & \mathbf{n}_3 &= \frac{1}{\sqrt{1 + \left(\frac{\lambda_1}{\gamma}\right)^2}}\left(-\frac{\lambda_1}{\gamma}\mathbf{i}_1 + \mathbf{i}_2\right)\ .
\end{aligned} \tag{1.215}$$

PROBLEM 1.21: Prove that the permutation symbol is a tensor of order three in Cartesian coordinate systems. (Hint: $e'_{k\ell r} = [\mathbf{i}'_k\,\mathbf{i}'_\ell\,\mathbf{i}'_r]$.)

PROBLEM 1.22: Prove the following relationship:

$$\frac{\partial \det(\mathbf{W})}{\partial \mathbf{W}} = \det(\mathbf{W})\,\mathbf{W}^{-T}\,. \tag{1.216}$$

CHAPTER 2

Kinematics

2.1. Deformation gradient

2.1.1. Configurations. The body $\mathcal{B}$ is a set of material points denoted by $\hat{P}$. The assignments of these points to a unique position in the 3D space is a configuration of the body. The region V° with boundary $A^\circ = \partial V^\circ$ the body occupies at time $t = t^\circ = 0$ is called reference configuration since the other configurations of the body are to be compared against the reference configuration. The region V with boundary $A = \partial V$ the body occupies at time t is the current (or deformed) configuration of the body. The initial configuration of the body

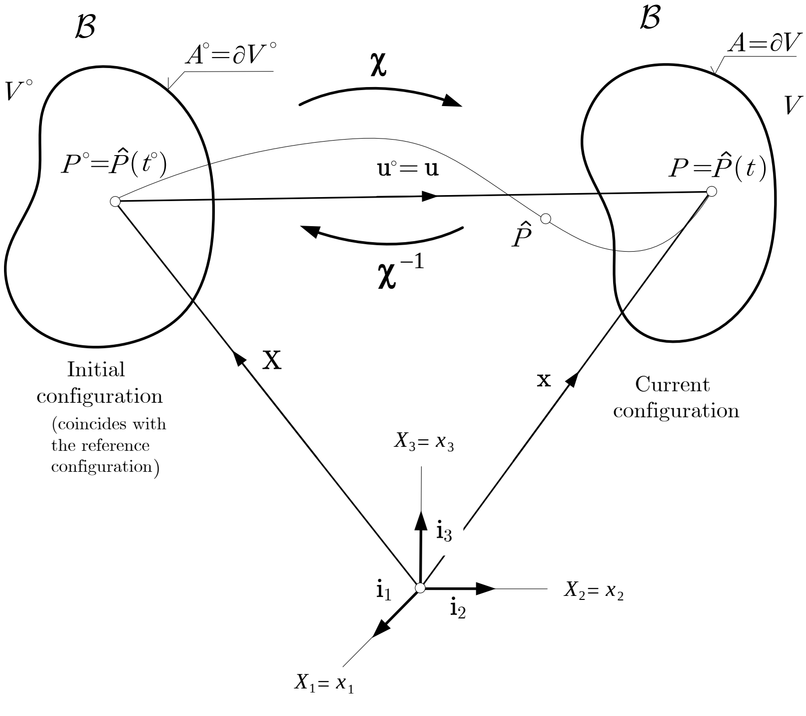

FIGURE 2.1. The body $\mathcal{B}$ in the initial and current configurations

is that of the body at time $t = t^\circ = 0$. We shall, therefore, assume that the initial configuration of the body coincides with the reference configuration. It might be, sometimes, advantageous to apply two different (in most cases curvilinear) coordinate systems, one for the initial configuration, the other for the current configuration. We shall, however, use the same Cartesian coordinate system for the two configurations. Distinction between the two configurations is made, in general, in two ways: (a) by typesetting a small circle as a superscript to a quantity regarded in the initial configuration or (b) by applying different notations in the two configurations: [capital (uppercase)] {small (lowercase)} letters will be used, in general, for denoting scalars, vectors, tensors and subscripts in the [initial] {current} configurations.

A material point (particle) is denoted by $\hat{P}$. In the initial (reference) configuration $\hat{P}(t^o) = P^o$, in the current configuration $\hat{P}(t) = P$.

The initial coordinates of a material point, i.e., its coordinates in the initial (reference) configuration are denoted by capital letters (lightface majuscules). For denoting the coordinates of the same material point in the current configuration, however, small letters (lightface minuscules) will be used. Indices (subscripts) in the [initial] (current) configuration are typeset, as has already been said, also in lightface [majuscules] (minuscules). Hence,

$$\mathbf{X} = X_A \, \mathbf{i}_A \quad \text{and} \quad \mathbf{x} = x_\ell \, \mathbf{i}_\ell \tag{2.1}$$

are the position vectors of the material point $\hat{P}$ in the initial and current configurations.

The coordinates X_1, X_2, X_3 identify the material point $\hat{P}$ in the initial configuration, therefore they are called material coordinates.

The coordinates x_1, x_2, x_3 show where the material point $\hat{P}$ can be found in space at time t, therefore they are called spatial coordinates.

Equation

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}; t) \,, \qquad x_\ell = \chi_\ell (X_1, X_2, X_3; t) \tag{2.2}$$

is the motion law: it gives the location of the material point $\hat{P}$ at time t, i.e, in the current configuration. Its inverse, i.e., the inverse motion law is of the form

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}; t) \,, \qquad X_A = \chi_A^{-1} (x_1, x_2, x_3; t) \,. \tag{2.3}$$

This equation shows where the material point (particle) P was when the motion began. We shall assume that relation (2.2) is one to one: then it has a unique inverse.

EXERCISE 2.1: Assume that the quasi-static law of motion $x_\ell = \chi_\ell(X_1, X_2, X_3)$ in the unit cube shown in Figure 2.2 is of the form

$$x_1 = X_1 + a_1 X_2^2\,, \quad x_2 = X_2 + a_2\,, \qquad x_3 = X_3 + a_3 X_2 X_3\,, \tag{2.4}$$

where a_1, a_2 and a_3 are positive constants. Find the inverse motion law.

If we solve the above equations for X_A in terms of x_ℓ we get

$$X_1 = x_1 - a_1\,(x_2 - a_2)^2, \quad X_2 = x_2 - a_2\,, \qquad X_3 = \frac{x_3}{1 + a_3(x_2 - a_2)}\,. \tag{2.5}$$

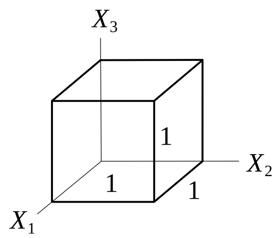

FIGURE 2.2. Unit cube

2.1.2. Material and spatial descriptions. Let $\boldsymbol{S}(P;t)$ be a tensor which represents a physical quantity, i.e., it describes a state of the material point $\hat{P}$ in the current configuration. This tensor can be given in two ways: (a) either as a function of the material coordinates X_1, X_2, X_3 and time t, (b) or as a function of the spatial coordinates x_1, x_2, x_3 and time t.

In the first case we speak about material description since the tensor $\boldsymbol{S}$ is attached mentally to the material point (particle) $\hat{P}$.

In the second case we speak about spatial description since the tensor $\boldsymbol{S}$ describes the spatial distribution of the physical quantity considered, i.e., it gives the physical state of that material point only which is at the spatial point x_1, x_2, x_3 at time t (which passes through the spatial point x_1, x_2, x_3 at time t).

It is noteworthy to mention that the two descriptions are equivalent. For solid bodies we more often use the material description. For gaseous and fluid continua, however, the spatial (or Eulerian[1]) description is more convenient.

For [material] {spatial} description the tensor function $\boldsymbol{S}$ is of the form

$$[\boldsymbol{S} = \boldsymbol{\Psi}\,(X_1, X_2, X_3; t)] \qquad \{\boldsymbol{S} = \boldsymbol{\Phi}\,(x_1, x_2, x_3; t)\}\,. \tag{2.6}$$

It is obvious that

$$\boldsymbol{\Phi}\,(x_1, x_2, x_3; t) = \boldsymbol{\Phi}\,[\chi_1\,(X_1, X_2, X_3; t)\,, \chi_2\,(X_1, X_2, X_3; t)\,, \chi_3\,(X_1, X_2, X_3; t)\,; t] = \\ = \boldsymbol{\Psi}\,(X_1, X_2, X_3; t)\,. \tag{2.7}$$

2.1.3. Displacement field. The displacement of a material point is defined by the following equations:

$$\mathbf{u}^\circ = \boldsymbol{\chi}(\mathbf{X}; t) - \mathbf{X}\,, \qquad u_A = \chi_A\,(X_1, X_2, X_3; t) - X_A; \tag{2.8a}$$

(material description)

$$\mathbf{u} = \mathbf{x} - \boldsymbol{\chi}^{-1}(\mathbf{x}; t)\,, \qquad u_\ell = x_\ell - \chi_\ell^{-1}\,(x_1, x_2, x_3; t)\,. \tag{2.8b}$$

(spatial description)

[1] Leonhard Euler (1707-1783)

It is obvious that

$$\mathbf{u}^\circ = \mathbf{u}\,. \tag{2.9}$$

In material description the displacement vector is attached to the initial position of the material point point $\hat{P}$ (to the point P°), in spatial description to the spatial point (x_1, x_2, x_3) where the material point $\hat{P}$ is at time t.

With $\mathbf{u}^\circ$ the motion law is of the form

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X};t) = \mathbf{X} + \mathbf{u}^\circ\,. \tag{2.10}$$

2.1.4. Deformation gradients. Equation $\boldsymbol{\kappa}^\circ(s^\circ)$ is that of a material line in the initial configuration – s^o is the arc coordinate measured on the material line considered. The material line $\boldsymbol{\kappa}^\circ(s^\circ)$ is deformed into the material line $\boldsymbol{\kappa}(s;t)$ of the current configuration, s is the arc coordinate along the deformed material line. Figure 2.3 shows, among others, these two material lines.

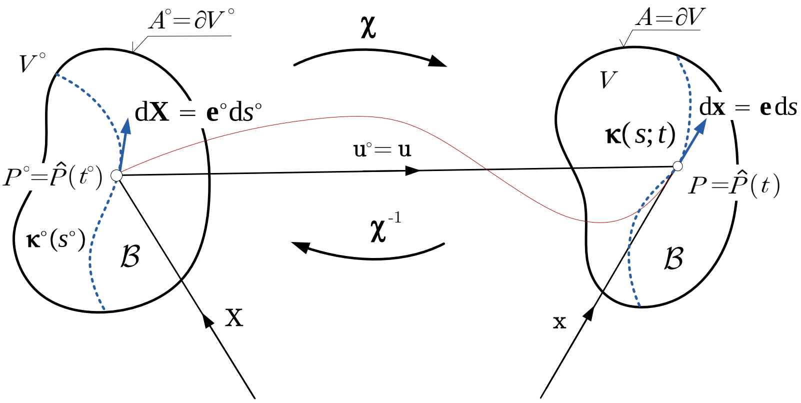

FIGURE 2.3. A material line before and after deformation s

The material line element vector (arc element vecror)

$$\mathrm{d}\mathbf{X} = \mathbf{e}^\circ \mathrm{d}s^\circ\,, \qquad \mathrm{d}X_A = e_A \mathrm{d}s^\circ\,, \qquad |\mathbf{e}^\circ| = 1 \tag{2.11a}$$

at the point P° on the material line $\boldsymbol{\kappa}^\circ(s^\circ)$ is deformed into the material line element vector (arc element vector)

$$\mathrm{d}\mathbf{x} = \mathbf{e}\,\mathrm{d}s\,, \qquad \mathrm{d}x_\ell = e_\ell \mathrm{d}s\,, \qquad |\mathbf{e}| = 1 \tag{2.11b}$$

at the point P on the material line $\boldsymbol{\kappa}(s;t)$. The two material line element vectors are related to each other via the following equation

$$\mathrm{d}\mathbf{x} = \frac{\partial \boldsymbol{\chi}}{\partial X_A}\mathrm{d}X_A = \left(\boldsymbol{\chi}\frac{\partial}{\partial X_A}\right)\underbrace{\mathbf{i}_A\cdot \mathrm{d}\mathbf{X}}_{\mathrm{d}X_A} = \left(\boldsymbol{\chi}\circ\underbrace{\frac{\partial}{\partial X_A}\mathbf{i}_A}_{\nabla^\circ}\right)\cdot \mathrm{d}\mathbf{X}\,, \tag{2.12}$$

where

$$\boxed{\boldsymbol{F} = \boldsymbol{\chi} \circ \nabla^{\circ} \underset{\boldsymbol{\chi}=\chi_\ell \mathbf{i}_\ell}{\uparrow} = \frac{\partial \chi_\ell}{\partial X_A} \mathbf{i}_\ell \circ \mathbf{i}_A = F_{\ell A} \mathbf{i}_\ell \circ \mathbf{i}_A \,, \qquad F_{\ell A} = \frac{\partial \chi_\ell}{\partial X_A}} \tag{2.13}$$

is the deformation gradient. Equation

$$\boxed{\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\mathbf{X} \,, \qquad \mathrm{d}x_\ell = F_{\ell A}\, \mathrm{d}X_A} \tag{2.14}$$

is a homogeneous linear relationship between the material line elements $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{X}$.

Consider now the inverse motion law, i.e., equation (2.3). We can write, in the same manner as we did for equation (2.12), that

$$\mathrm{d}\mathbf{X} = \frac{\partial \boldsymbol{\chi}^{-1}}{\partial x_k} \mathrm{d}x_k = \left(\boldsymbol{\chi}^{-1} \frac{\partial}{\partial x_k} \right) \underbrace{\mathbf{i}_k \cdot \mathrm{d}\mathbf{x}}_{\mathrm{d}x_k} = \left(\boldsymbol{\chi}^{-1} \circ \underbrace{\frac{\partial}{\partial x_k} \mathbf{i}_k}_{\nabla} \right) \cdot \mathrm{d}\mathbf{x} \,, \tag{2.15}$$

where

$$\boxed{\boldsymbol{F}^{-1} = \boldsymbol{\chi}^{-1} \circ \nabla \underset{\boldsymbol{\chi}^{-1}=\chi_B^{-1} \mathbf{i}_B}{\uparrow} = \frac{\partial \chi_B^{-1}}{\partial x_k} \mathbf{i}_B \circ \mathbf{i}_k = F_{Bk}^{-1} \mathbf{i}_B \circ \mathbf{i}_k \,, \quad F_{Bk}^{-1} = \frac{\partial \chi_B^{-1}}{\partial x_k}} \tag{2.16}$$

is the inverse deformation gradient. Equation

$$\boxed{\mathrm{d}\mathbf{X} = \boldsymbol{F}^{-1} \cdot \mathrm{d}\mathbf{x} \,, \qquad \mathrm{d}X_B = F_{Bk}^{-1}\, \mathrm{d}x_k} \tag{2.17}$$

relates $\mathrm{d}\mathbf{X}$ to $\mathrm{d}\mathbf{x}$. Upon substitution of (2.12) into $(2.17)_1$ we get

$$\mathrm{d}\mathbf{X} = \boldsymbol{F}^{-1} \cdot \boldsymbol{F} \cdot \mathrm{d}\mathbf{X} \,, \tag{2.18}$$

which can be satisfied for any $\mathrm{d}\mathbf{X}$ if and only if

$$\boldsymbol{F}^{-1} \cdot \boldsymbol{F} = \boldsymbol{F} \cdot \boldsymbol{F}^{-1} = \boldsymbol{1} \,. \tag{2.19}$$

REMARK 2.1: The deformation gradients $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ are two point tensors since [χ_ℓ and X_A in $(2.13)_2$] {χ_B^{-1} and x_k in $(2.17)_2$} belong to the points [P and P°] {P° and P}.

REMARK 2.2: Equation (2.14) is one to one relationship between $\mathrm{d}x_\ell$ and $\mathrm{d}X_A$ if

$$\boxed{J = \det(\boldsymbol{F}) = e_{PQR} F_{1P} F_{2Q} F_{3R} = \begin{vmatrix} \frac{\partial \chi_1}{\partial X_1} & \frac{\partial \chi_1}{\partial X_2} & \frac{\partial \chi_1}{\partial X_3} \\ \frac{\partial \chi_2}{\partial X_1} & \frac{\partial \chi_2}{\partial X_2} & \frac{\partial \chi_2}{\partial X_3} \\ \frac{\partial \chi_3}{\partial X_1} & \frac{\partial \chi_3}{\partial X_2} & \frac{\partial \chi_3}{\partial X_3} \end{vmatrix} \neq 0 \,,} \tag{2.20}$$

where $J = \det(\boldsymbol{F})$ is the Jacobian[2]. It is proven in Subsection 6.2.1 of Chapter 6, which is devoted to the principle of mass conservation, that $J > 0$. It is obvious that

$$\det(\boldsymbol{F}^{-1}) = \frac{1}{J} \,. \tag{2.21}$$

[2]Named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851)

Making use of the equations

$$\mathbf{X} \circ \nabla^{\circ} = \mathbf{X} \circ \frac{\partial}{\partial X_B}\mathbf{i}_B = \frac{\partial X_A}{\partial X_B}\mathbf{i}_A \circ \mathbf{i}_B = \delta_{AB}\mathbf{i}_A \circ \mathbf{i}_B = \boldsymbol{1}\,, \tag{2.22a}$$

$$\mathbf{x} \circ \nabla = \mathbf{x} \circ \frac{\partial}{\partial x_\ell}\mathbf{i}_\ell = \frac{\partial x_k}{\partial x_\ell}\mathbf{i}_k \circ \mathbf{i}_\ell = \delta_{k\ell}\mathbf{i}_k \circ \mathbf{i}_\ell = \boldsymbol{1} \tag{2.22b}$$

(the gradient of the position vector is the unit tensor) and the relations

$$\boldsymbol{\chi} = \mathbf{X} + \mathbf{u}^{\circ}\,, \qquad \boldsymbol{\chi}^{-1} = \mathbf{x} - \mathbf{u} \tag{2.23}$$

we obtain the deformation gradients in terms of $\mathbf{u}^{\circ} = \mathbf{u}$:

$$\boxed{\begin{gathered} \boldsymbol{F} = \boldsymbol{\chi} \circ \nabla^{\circ} = \mathbf{X} \circ \nabla^{\circ} + \mathbf{u}^{\circ} \circ \nabla^{\circ} = \boldsymbol{1} + \mathbf{u}^{\circ} \circ \nabla^{\circ}\,, \\ F_{AB} = \delta_{AB} + u_A \nabla_B = \delta_{AB} + u_{A,B}\,; \end{gathered}} \tag{2.24}$$

$$\boxed{\begin{gathered} \boldsymbol{F}^{-1} = \boldsymbol{\chi}^{-1} \circ \nabla = \mathbf{x} \circ \nabla - \mathbf{u} \circ \nabla = \boldsymbol{1} - \mathbf{u} \circ \nabla\,, \\ F^{-1}_{k\ell} = \delta_{k\ell} - u_k \nabla_\ell = \delta_{k\ell} - u_{k,\ell}\,. \end{gathered}} \tag{2.25}$$

Equation [(2.24)] {(2.25)} gives the components of the [deformation gradient] {inverse deformation gradient} in the [inital] {current} configuration of the continuum.

Utilizing equations (1.105) and (1.106) we can find the inverse deformation gradient in the initial configuration:

$$\begin{aligned} F^{-1}_{PL} &= \frac{1}{2}\frac{e_{PQR}e_{LJK}F_{JQ}F_{KR}}{|F_{MN}|} = \\ &= \frac{1}{J}\underbrace{\frac{1}{2}e_{PQR}e_{LJK}\left(\delta_{JQ}+u^{\circ}_{J,Q}\right)\left(\delta_{KR}+u^{\circ}_{K,R}\right)}_{\mathcal{F}_{PL}} = \frac{1}{J}\mathcal{F}_{PL}\,. \end{aligned} \tag{2.26}$$

This equation can be also be given in direct notation:

$$\boldsymbol{F}^{-1} = \frac{1}{J}\boldsymbol{\mathcal{F}}\,, \qquad \boldsymbol{\mathcal{F}} = \mathcal{F}_{PL}\mathbf{i}_P \circ \mathbf{i}_L\,. \tag{2.27}$$

Exercise 2.2: What are the Cartesian components of $\boldsymbol{F}$ in terms of the Cartesian displacement components?

It follows from (2.24) that

$$\underset{(3\times3)}{\underline{\mathbf{F}}} = [F_{AB}] = \begin{bmatrix} 1 + \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & 1 + \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & 1 + \frac{\partial u_3}{\partial X_3} \end{bmatrix}. \tag{2.28}$$

2.1.5. Relations between gradients taken in the initial and current configurations. Let

$$\varPhi(x_1,x_2,x_3)=\varPhi[x_1(X_1,X_2,X_3;t),x_2(X_1,X_2,X_3;t),x_3(X_1,X_2,X_3;t)]= \\ =\varPhi(X_1,X_2,X_3;t)$$

be a scalar field regarded in the current and initial configurations. Our aim is to clarify how the gradient $\varPhi\nabla$, which describes the local changes in a small neighborhood of the spacial point P, and the gradient $\varPhi\nabla^\circ$, which describes the local changes of the same scalar field in the neighborhood of the point P°, i.e., in the initial configuration, are related to each other. If we apply the chain rule we can write

$$\varPhi\,\nabla^\circ=\varPhi\frac{\partial}{\partial X_A}\mathbf{i}_A=\frac{\partial\varPhi}{\partial x_\ell}\frac{\partial x_\ell}{\partial X_A}\mathbf{i}_A=\underset{(2.13)}{\uparrow}=\frac{\partial\varPhi}{\partial x_\ell}F_{\ell A}\mathbf{i}_A= \\ =\frac{\partial\varPhi}{\partial x_k}\delta_{k\ell}\,F_{\ell A}\mathbf{i}_A=\overset{\downarrow}{\varPhi}\frac{\partial}{\partial x_k}\mathbf{i}_k\cdot(F_{\ell A}\mathbf{i}_\ell\circ\mathbf{i}_A)=\overset{\downarrow}{\varPhi}\,\nabla\cdot\boldsymbol{F}$$

or

$$\boxed{\varPhi\,\nabla^\circ=\overset{\downarrow}{\varPhi}\,\nabla\cdot\boldsymbol{F}\qquad\text{and}\qquad\varPhi\,\nabla=\overset{\downarrow}{\varPhi}\,\nabla^\circ\cdot\boldsymbol{F}^{-1}\,,}\tag{2.29a}$$

where the down arrow shows the quantity the operator ∇ should be applied to.

Dot multiply equation (2.25) from right by $\boldsymbol{F}$ and substitute then expression (2.24). We obtain

$$\boldsymbol{1}=\boldsymbol{F}^{-1}\cdot\boldsymbol{F}=\boldsymbol{F}-(\mathbf{u}\circ\nabla)\cdot\boldsymbol{F}=\boldsymbol{1}+\mathbf{u}^\circ\circ\nabla^\circ-(\mathbf{u}\circ\nabla)\cdot\boldsymbol{F}$$

from where it follows that

$$\boxed{\mathbf{u}^\circ\circ\nabla^\circ=(\mathbf{u}\circ\nabla)\cdot\boldsymbol{F}\qquad\text{and}\qquad\mathbf{u}\circ\nabla=(\mathbf{u}^\circ\circ\nabla^\circ)\cdot\boldsymbol{F}^{-1}\,.}\tag{2.29b}$$

Equations $(2.29a)_2$ and $(2.29b)_2$ can easily be obtained from $(2.29a)_1$ and $(2.29b)_1$ if we dot multiply the later by $\boldsymbol{F}^{-1}$. According to (2.29) it also holds that

$$\nabla^\circ=\nabla\cdot\boldsymbol{F}\ \text{ and }\ \nabla=\nabla^\circ\cdot\boldsymbol{F}^{-1}\ \text{ or }\ \nabla^\circ_A=\nabla_\ell F_{\ell A}\ \text{ and }\ \nabla_\ell=\nabla^\circ_A\,F^{-1}_{A\ell}\,.\tag{2.30}$$

2.2. Strain tensors

2.2.1. Strain tensors in the initial configuration. Making use of equations (2.11), (2.14) and (2.17) we may write

$$\mathrm{d}s^2-(\mathrm{d}s^\circ)^2=\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}-\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X}=(\boldsymbol{F}\cdot\mathrm{d}\mathbf{X})^2-\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X}= \\ =\mathrm{d}\mathbf{X}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot\mathrm{d}\mathbf{X}-\mathrm{d}\mathbf{X}\cdot\boldsymbol{1}\cdot\mathrm{d}\mathbf{X}=2\,(\mathrm{d}s^\circ)^2\,\mathbf{e}^\circ\cdot\frac{1}{2}\left(\boldsymbol{F}^T\cdot\boldsymbol{F}-\boldsymbol{1}\right)\cdot\mathbf{e}^\circ\,.\tag{2.31}$$

On the base of the above relationship we define the right Cauchy-Green[3] tensor (right Cauchy-Green deformation tensor) $\boldsymbol{C}$ [9, 5] and the Green-Lagrange[4]

[3] Augustin Cauchy, 1789-1857; George Green, 1793-1841
[4] Joseph Loius Lagrange, 1736-1813

strain tensor $\boldsymbol{E}$ by the following equations:

$$\boxed{\begin{aligned} \boldsymbol{C} &= \boldsymbol{F}^T \cdot \boldsymbol{F}\,, & \boldsymbol{E} &= \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{1}\right)\,, \\ C_{AB} &= F_{Am} F_{mB}\,, & E_{AB} &= \frac{1}{2}\left(C_{AB} - \delta_{AB}\right)\,. \end{aligned}} \tag{2.32}$$

Both the right Cauchy-Green tensor $\boldsymbol{C}$ and the Green-Lagrange strain tensor $\boldsymbol{E}$ belong to the initial configuration of the body.

REMARK 2.3: The right Cauchy-Green tensor $\boldsymbol{C}$ as well as the Green-Lagrange strain tensor $\boldsymbol{E}$ are clearly symmetric:

$$\left.\begin{aligned} \boldsymbol{C} &= \boldsymbol{C}^T\,, & \boldsymbol{E} &= \boldsymbol{E}^T\,, \\ C_{AB} &= C_{BA} & E_{AB} &= E_{BA}\,. \end{aligned}\right\} \tag{2.33}$$

REMARK 2.4: The inequality

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{X} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{X} \cdot \boldsymbol{C} \cdot \mathrm{d}\mathbf{X} > 0 \tag{2.34}$$

holds for any $\mathrm{d}\mathbf{X} \neq 0$. Thus the right Cauchy-Green tensor is positive definite.

REMARK 2.5: Upon substitution of equation (2.11a) into (2.34) we get

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{X} \cdot \boldsymbol{C} \cdot \mathrm{d}\mathbf{X} = \left(\mathrm{d}s^\circ\right)^2 \mathbf{e}^\circ \cdot \boldsymbol{C} \cdot \mathbf{e}^\circ$$

from where

$$\lambda^e = \frac{\mathrm{d}s}{\mathrm{d}s^\circ} = \sqrt{\mathbf{e}^\circ \cdot \boldsymbol{C} \cdot \mathbf{e}^\circ}\,. \tag{2.35}$$

Here λ^e is the stretch ratio (or stretch for short).

REMARK 2.6: The axial (or normal) strain is the change in length per initial length for a material line element:

$$\begin{aligned} \varepsilon^{\circ\, e} &= \frac{\mathrm{d}s - \mathrm{d}s^\circ}{\mathrm{d}s^\circ} = \lambda^e - 1 = \underset{(2.34)}{\uparrow} = \sqrt{\mathbf{e}^\circ \cdot \boldsymbol{C} \cdot \mathbf{e}^\circ} - 1 = \underset{\boldsymbol{C} = \boldsymbol{1} + 2\boldsymbol{E}}{\uparrow} = \\ &= \sqrt{\mathbf{e}^\circ \cdot (\boldsymbol{1} + 2\boldsymbol{E}) \cdot \mathbf{e}^\circ} - 1 = \sqrt{1 + 2\mathbf{e}^\circ \cdot \boldsymbol{E} \cdot \mathbf{e}^\circ} - 1\,. \end{aligned} \tag{2.36}$$

It is also customary to define the axial strain as the change in length per the length of a material line element in the current configuration:

$$\varepsilon^e = \frac{\mathrm{d}s - \mathrm{d}s^\circ}{\mathrm{d}s} = 1 - \frac{1}{\lambda^e} = \frac{\lambda^e - 1}{\lambda^e} = \frac{\varepsilon^{\circ\, e}}{\lambda^e}\,. \tag{2.37}$$

The axial strain is either positive or negative depending on whether the material line element experiences tension or contraction. If the length of the material line element does not change then the axial strain is zero.

REMARK 2.7: Assume that $|\mathbf{e}^\circ \cdot \boldsymbol{E} \cdot \mathbf{e}^\circ| \ll 1$ for any $\mathbf{e}^\circ$. Utilizing the estimation $\sqrt{1 \pm 2x} \approx 1 \pm x$ if $|x| \ll 1$ we obtain from (2.36) that

$$\varepsilon^{\circ\, e} = \mathbf{e}^\circ \cdot \boldsymbol{\varepsilon}^\circ \cdot \mathbf{e}^\circ \tag{2.38}$$

in which we applied the notation: $\boldsymbol{\varepsilon}^\circ = \boldsymbol{E}$ if $|\mathbf{e}^\circ \cdot \boldsymbol{E} \cdot \mathbf{e}^\circ| \ll 1$.

The Green-Lagrange strain tensor can be given in terms of the displacement vector. To this end substitute (2.24) into (2.32) to get

$$\boldsymbol{E}=\frac{1}{2}\left(\boldsymbol{C}-\boldsymbol{1}\right)=\frac{1}{2}\left(\boldsymbol{F}^T\cdot\boldsymbol{F}-\boldsymbol{1}\right)=\frac{1}{2}\left[\left(\boldsymbol{1}+\nabla^\circ\circ\mathbf{u}^\circ\right)\cdot\left(\boldsymbol{1}+\mathbf{u}^\circ\circ\nabla^\circ\right)-\boldsymbol{1}\right]$$

from where it follows that

$$\boxed{\begin{aligned}\boldsymbol{E}&=\frac{1}{2}\left[\mathbf{u}^\circ\circ\nabla^\circ+\nabla^\circ\circ\mathbf{u}^\circ+\left(\nabla^\circ\circ\mathbf{u}^\circ\right)\cdot\left(\mathbf{u}^\circ\circ\nabla^\circ\right)\right]\,,\\ E_{AB}&=\frac{1}{2}\left[u_A\nabla_B+\nabla_A u_B+\left(\nabla_A u_K\right)\left(u_K\nabla_B\right)\right]\,,\\ E_{AB}&=\frac{1}{2}\left[u_{A,B}+u_{B,A}+u_{K,A}u_{K,B}\right]\,.\end{aligned}}\tag{2.39}$$

REMARK 2.8: Assume that $|u_{A,B}|\ll 1$. Then the quadratic term can be neglected:

$$\boxed{\boldsymbol{\varepsilon}^\circ=\frac{1}{2}\left(\mathbf{u}^\circ\circ\nabla^\circ+\nabla^\circ\circ\mathbf{u}^\circ\right)\,,\qquad \varepsilon_{AB}=\frac{1}{2}\left(u_{A,B}+u_{B,A}\right)\,.}\tag{2.40}$$

Note that we have applied the same notation for the strain tensor here as in Remark 2.7. The reason for this is very simple: if $|u_{A,B}|\ll 1$ then it also holds that $\mathbf{e}^\circ\cdot\boldsymbol{E}\cdot\mathbf{e}^\circ\ll 1$.

REMARK 2.9: Let $\mathrm{d}\mathbf{X}_I$, $\mathrm{d}\mathbf{x}_I$ and $\mathrm{d}\mathbf{X}_{II}$, $\mathrm{d}\mathbf{x}_{II}$ be two different arc element vectors – see Figure 2.4. The angles formed by $\mathrm{d}\mathbf{X}_I$, $\mathrm{d}\mathbf{X}_{II}$ and $\mathrm{d}\mathbf{x}_I$, $\mathrm{d}\mathbf{x}_{II}$ will be denoted

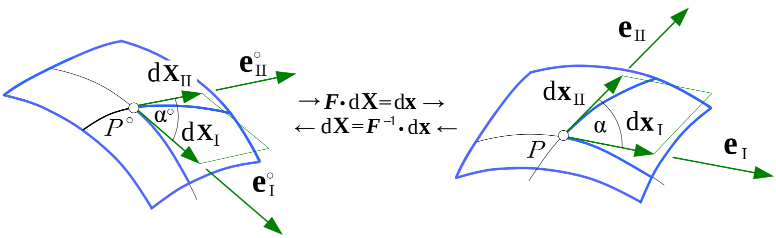

FIGURE 2.4. Angle change

by α° and α. It is obvious that

$$\begin{aligned}\mathrm{d}\mathbf{X}_I&=\mathbf{e}^\circ_I\mathrm{d}s^\circ_I\,, & \mathrm{d}X^I_A&=e^I_A\mathrm{d}s^\circ_I\,,\\ \mathrm{d}\mathbf{X}_{II}&=\mathbf{e}^\circ_{II}\mathrm{d}s^\circ_{II}\,, & \mathrm{d}X^{II}_A&=e^{II}_A\mathrm{d}s^\circ_{II}\end{aligned}\tag{2.41a}$$

and

$$\begin{aligned}\mathrm{d}\mathbf{x}_I&=\mathbf{e}_I\mathrm{d}s_I\,, & \mathrm{d}x^I_k&=e^I_k\mathrm{d}s_I\,,\\ \mathrm{d}\mathbf{x}_{II}&=\mathbf{e}_{II}\mathrm{d}s_{II}\,, & \mathrm{d}x^{II}_k&=e^{II}_k\mathrm{d}s_{II}\,.\end{aligned}\tag{2.41b}$$

With these arc element vectors we get

$$\begin{aligned}\mathrm{d}\mathbf{x}_I\cdot\mathrm{d}\mathbf{x}_{II}-\mathrm{d}\mathbf{X}_I\cdot\mathrm{d}\mathbf{X}_{II}&=\left(\boldsymbol{F}\cdot\mathrm{d}\mathbf{X}_I\right)\cdot\left(\boldsymbol{F}\cdot\mathrm{d}\mathbf{X}_{II}\right)-\mathrm{d}\mathbf{X}_I\cdot\mathrm{d}\mathbf{X}_{II}=\\ &=\mathrm{d}\mathbf{X}_I\cdot\left(\boldsymbol{F}^T\cdot\boldsymbol{F}-\boldsymbol{1}\right)\cdot\mathrm{d}\mathbf{X}_{II}=2\mathrm{d}s^\circ_I\mathrm{d}s^\circ_{II}\mathbf{e}^\circ_I\cdot\boldsymbol{E}\cdot\mathbf{e}^\circ_{II}\,.\end{aligned}\tag{2.42a}$$

On the other hand

$$\mathrm{d}\mathbf{x}_I \cdot \mathrm{d}\mathbf{x}_{II} - \mathrm{d}\mathbf{X}_I \cdot \mathrm{d}\mathbf{X}_{II} = \mathrm{d}s_I \mathrm{d}s_{II} \cos\alpha - \mathrm{d}s_I^\circ \mathrm{d}s_{II}^\circ \cos\alpha^\circ\,. \tag{2.42b}$$

Hence

$$\boxed{\frac{\mathrm{d}s_I}{\mathrm{d}s_I^\circ}\frac{\mathrm{d}s_{II}}{\mathrm{d}s_{II}^\circ}\cos\alpha - \cos\alpha^\circ = \lambda_I^e \lambda_{II}^e \cos\alpha - \cos\alpha^\circ = 2\mathbf{e}_I^\circ \cdot \boldsymbol{E} \cdot \mathbf{e}_{II}^\circ\,.} \tag{2.43}$$

This equation makes possible to calculate the angle α.

Assume that $\alpha^\circ = \pi/2$ and $\alpha = \alpha^\circ - \gamma_{12}$ where γ_{12} is the angle change for the angle $\alpha^\circ = \pi/2$. Then we get from (2.43) that

$$\cos\left(\pi/2 - \gamma_{12}\right) = \sin\gamma_{12} = \frac{2\mathbf{e}_I^\circ \cdot \boldsymbol{E} \cdot \mathbf{e}_{II}^\circ}{\lambda_I^e \lambda_{II}^e} = \frac{2\mathbf{e}_I^\circ \cdot \boldsymbol{E} \cdot \mathbf{e}_{II}^\circ}{(1+\varepsilon_I^{\circ\, e})\,(1+\varepsilon_{II}^{\circ\, e})}\,. \tag{2.44}$$

If the deformations are small $\sin\gamma_{12} \approx \gamma_{12}$, $1+\varepsilon_I^{\circ\, e} \approx 1$, $1+\varepsilon_{II}^{\circ\, e} \approx 1$, $\boldsymbol{E} = \boldsymbol{\varepsilon}^\circ$. Consequently,

$$\boxed{\gamma_{12} = 2\mathbf{e}_I^\circ \cdot \boldsymbol{\varepsilon}^\circ \cdot \mathbf{e}_{II}^\circ\,.} \tag{2.45}$$

REMARK 2.10: The inverse $\boldsymbol{C}^{-1}$ is the Piola[5] strain tensor:

$$C_{AB}^{-1} = F_{Am}^{-1} F_{mB}^{-1}\,. \tag{2.46}$$

2.2.2. Strain tensors in the current configuration. If we utilize equation (2.17) we may rewrite (2.34) into the following form:

$$\begin{aligned} \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} - \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X} &= \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} - \left(\boldsymbol{F}^{-1}\cdot\mathrm{d}\mathbf{x}\right)\cdot\left(\boldsymbol{F}^{-1}\cdot\mathrm{d}\mathbf{x}\right) = \\ &= \mathrm{d}\mathbf{x}\cdot\left(\boldsymbol{1} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right)\cdot\mathrm{d}\mathbf{x} = 2\,(\mathrm{d}s)^2\,\mathbf{e}\cdot\frac{1}{2}\left(\boldsymbol{1} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right)\cdot\mathbf{e}\,. \end{aligned} \tag{2.47}$$

On the basis of the above equation we define the left Cauchy-Green tensor (left Cauchy-Green deformation tensor) $\boldsymbol{b}^{-1}$ and the Euler-Almansi[6] strain tensor $\boldsymbol{e}$ by the following relations [36, 37, 38]:

$$\boxed{\begin{aligned} \boldsymbol{b}^{-1} &= \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\,, & \boldsymbol{e} &= \frac{1}{2}\left(\boldsymbol{1} - \boldsymbol{b}^{-1}\right)\,, \\ b_{k\ell}^{-1} &= F_{kA}^{-1} F_{A\ell}^{-1}\,, & e_{k\ell} &= \frac{1}{2}\left(\delta_{k\ell} - b_{k\ell}^{-1}\right)\,. \end{aligned}} \tag{2.48}$$

REMARK 2.11: The tensor

$$\boldsymbol{b} = \boldsymbol{F}\cdot\boldsymbol{F}^T\,, \qquad b_{pq} = F_{pA}F_{Aq} \tag{2.49}$$

is the Cauchy strain tensor.

REMARK 2.12: The left Cauchy-Green tensor $\boldsymbol{b}^{-1}$, the Cauchy strain tensor $\boldsymbol{b}$ and the Euler-Almansi strain tensor $\boldsymbol{e}$ are all symmetric tensors:

$$\left.\begin{aligned} \boldsymbol{b}^{-1} &= \boldsymbol{b}^{-T}\,, & \boldsymbol{b} &= \boldsymbol{b}^T\,, & \boldsymbol{e} &= \boldsymbol{e}^T\,, \\ b_{k\ell}^{-1} &= b_{\ell k}^{-1}\,, & b_{pq} &= b_{qp}\,, & e_{k\ell} &= e_{\ell k}\,. \end{aligned}\right\} \tag{2.50}$$

[5]Gabrio Piola, 1794-1850

[6]Emilio Almansi, 1869-1948

Remark 2.13: The inequality

$$(\mathrm{d}s^{\circ})^{2} = \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{x}\cdot\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\cdot\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x}\cdot\boldsymbol{b}^{-1}\cdot\mathrm{d}\mathbf{x} > 0 \tag{2.51}$$

holds for any $\mathrm{d}\mathbf{x} \neq 0$. Thus the left Cauchy-Green tensor $\boldsymbol{b}^{-1}$ is positive definite. If $\boldsymbol{b}^{-1}$ is positive definite then so is the Cauchy strain tensor $\mathbf{b}$.

Remark 2.14: As regards the stretch ratio we can write

$$(\mathrm{d}s^{\circ})^{2} = \mathrm{d}\mathbf{x}\cdot\boldsymbol{b}^{-1}\cdot\mathrm{d}\mathbf{x} = \mathrm{d}s^{2}\,\mathbf{e}\cdot\boldsymbol{b}^{-1}\cdot\mathbf{e}\,.$$

Hence

$$\lambda^{e} = \frac{\mathrm{d}s}{\mathrm{d}s^{\circ}} = \frac{1}{\sqrt{\mathbf{e}\cdot\boldsymbol{b}^{-1}\cdot\mathbf{e}}}\,. \tag{2.52}$$

Remark 2.15: With (2.52) the definition of the axial strain (2.36) yields

$$\varepsilon^{\circ\, e} = \lambda^{e} - 1 = \frac{1}{\sqrt{\mathbf{e}\cdot\boldsymbol{b}^{-1}\cdot\mathbf{e}}} - 1 = \underset{\boldsymbol{b}^{-1}=\boldsymbol{1}-2\boldsymbol{e}}{\uparrow} =$$

$$= \frac{1}{\sqrt{\mathbf{e}\cdot(\boldsymbol{1}-2\boldsymbol{e})\cdot\mathbf{e}}} - 1 = \frac{1}{\sqrt{1-2\mathbf{e}\cdot\boldsymbol{e}\cdot\mathbf{e}}} - 1\,. \tag{2.53}$$

Remark 2.16: Assume that $|\mathbf{e}\cdot\boldsymbol{e}\cdot\mathbf{e}| \ll 1$. Making use of the estimation

$$1/\sqrt{1\pm 2x} \approx 1 \mp x \ \text{ if } |x| \ll 1$$

we obtain from (2.53) that

$$\varepsilon^{\circ\, e} = \mathbf{e}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{e}\,, \tag{2.54}$$

where we applied the notation: $\boldsymbol{\varepsilon} = \boldsymbol{e}$ if $|\mathbf{e}\cdot\boldsymbol{e}\cdot\mathbf{e}| \ll 1$.

The Euler-Almansi strain tensor (likewise the Green-Lagrange strain tensor) can also be given in terms of the displacement vector. To this end substitute (2.25) into (2.48) to get

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{1} - \boldsymbol{b}^{-1}\right) = \frac{1}{2}\left(\boldsymbol{1} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right) = \frac{1}{2}\left[\boldsymbol{1} - (\boldsymbol{1} - \nabla\circ\mathbf{u})\cdot(\boldsymbol{1} - \mathbf{u}\circ\nabla)\right]$$

from where we have

$$\boxed{\begin{aligned} \boldsymbol{e} &= \frac{1}{2}\left[\mathbf{u}\circ\nabla + \nabla\circ\mathbf{u} - (\nabla\circ\mathbf{u})\cdot(\mathbf{u}\circ\nabla)\right], \\ e_{k\ell} &= \frac{1}{2}\left[u_k\nabla_\ell + \nabla_k u_\ell - (\nabla_k u_m)(u_m\nabla_\ell)\right], \\ e_{k\ell} &= \frac{1}{2}\left[u_{k,\ell} + u_{\ell,k} - u_{m,k}u_{m,\ell}\right]. \end{aligned}} \tag{2.55}$$

Remark 2.17: Assume that $|u_{k,\ell}| \ll 1$. Then the quadratic term can be neglected. Thus we get

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{u}\circ\nabla + \nabla\circ\mathbf{u})\,, \qquad \varepsilon_{k\ell} = \frac{1}{2}(u_{k,\ell} + u_{\ell,k})\,. \tag{2.56}$$

Note that we have applied the same notation for the strain tensor here as in Remark 2.16. The reason for this is very simple: if $|u_{k,\ell}| \ll 1$ then it also holds that $\mathbf{e}\cdot\boldsymbol{e}\cdot\mathbf{e} \ll 1$.

EXERCISE 2.3: Determine the angle change using the equations that are valid in spatial description.

On the basis of Figure 2.4 we can write

$$\begin{aligned} \mathrm{d}\mathbf{x}_I \cdot \mathrm{d}\mathbf{x}_{II} - \mathrm{d}\mathbf{X}_I \cdot \mathrm{d}\mathbf{X}_{II} &= \mathrm{d}\mathbf{x}_I \cdot \mathrm{d}\mathbf{x}_{II} - \left(\boldsymbol{F}^{-1} \cdot \mathrm{d}\mathbf{x}_I\right) \cdot \left(\boldsymbol{F}^{-1} \cdot \mathrm{d}\mathbf{x}_{II}\right) = \\ &= 2\,\mathrm{d}\mathbf{x}_I \cdot \frac{1}{2}\left(\boldsymbol{1} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}\right) \cdot \mathrm{d}\mathbf{x}_{II} = 2\mathrm{d}s_I \mathrm{d}s_{II} \mathbf{e}_I \cdot \boldsymbol{e} \cdot \mathbf{e}_{II}\,. \end{aligned} \tag{2.57a}$$

On the other hand

$$\mathrm{d}\mathbf{x}_I \cdot \mathrm{d}\mathbf{x}_{II} - \mathrm{d}\mathbf{X}_I \cdot \mathrm{d}\mathbf{X}_{II} = \mathrm{d}s_I \mathrm{d}s_{II} \cos\alpha - \mathrm{d}s_I^\circ \mathrm{d}s_{II}^\circ \cos\alpha^\circ\,. \tag{2.57b}$$

A comparison of equations (2.57a) and (2.57a) yields

$$\cos\alpha - \frac{\mathrm{d}s_I^\circ}{\mathrm{d}s_I}\frac{\mathrm{d}s_{II}^\circ}{\mathrm{d}s_{II}}\cos\alpha^\circ = \cos\alpha - \frac{1}{\lambda_I^e}\frac{1}{\lambda_{II}^e}\cos\alpha^\circ = 2\mathbf{e}_I \cdot \boldsymbol{e} \cdot \mathbf{e}_{II}\,. \tag{2.58}$$

This equation makes possible again to calculate the angle α.

If $\alpha = \alpha^\circ - \gamma_{12}$ where $\alpha^\circ = \pi/2$ and γ_{12} is the angle change we have

$$\cos\alpha = \cos\left(\pi/2 - \gamma_{12}\right) = \sin\gamma_{12} = 2\mathbf{e}_I \cdot \boldsymbol{e} \cdot \mathbf{e}_{II} \tag{2.59}$$

EXERCISE 2.4: Find the left Cauchy-Green tensor $\boldsymbol{b}^{-1}$ and the Euler-Almansi strain tensor $\boldsymbol{e}$ for the motion given in Exercise 2.1.

The inverse deformation gradient is given by equation (2.98). If we substitute it into the definition (2.48) of the left Cauchy-Green tensor we have

$$\underset{(3\times3)}{\underline{\boldsymbol{b}}}^{-1} = \left[b_{k\ell}^{-1}\right] = \left[b_{k\ell}^{-1}\right]\left[F_{kA}^{-1}F_{A\ell}^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 \\ -2a_1\left(x_2 - a_2\right) & 1 & \dfrac{-a_3x_3}{\left[1 + a_3\left(x_2 - a_2\right)\right]^2} \\ 0 & 0 & \dfrac{1}{1 + a_3\left(x_2 - a_2\right)} \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & -2a_1\left(x_2 - a_2\right) & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{-a_3x_3}{\left[1 + a_3\left(x_2 - a_2\right)\right]^2} & \dfrac{1}{1 + a_3\left(x_2 - a_2\right)} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2a_1\left(x_2 - a_2\right) & 1 + 4a_1^2\left(x_2 - a_2\right)^2 + \dfrac{a_3^2x_3^2}{\left[1 + a_3\left(x_2 - a_2\right)\right]^4} & \dfrac{-a_3x_3}{\left[1 + a_3\left(x_2 - a_2\right)\right]^3} \\ 0 & \dfrac{-a_3x_3}{\left[1 + a_3\left(x_2 - a_2\right)\right]^3} & \dfrac{1}{\left[1 + a_3\left(x_2 - a_2\right)\right]^2} \end{bmatrix}. \tag{2.60}$$

2.2.3. Eigenvalue problems, principal stretches. It is obvious on the basis of equation (2.34) that the square of the stretch ratios can be given in terms of the tensor $\boldsymbol{C}$, which belongs to the initial configuration. It follows from equation (2.52) that the reciprocals of these quantities can be given in terms $\boldsymbol{b}^{-1}$ which, in contrast to $\boldsymbol{C}$, is a tensor of the current configuration:

$$(\lambda^e)^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}s^\circ}\right)^2 = \mathbf{e}^\circ \cdot \boldsymbol{C} \cdot \mathbf{e}^\circ\,, \qquad \frac{1}{(\lambda^e)^2} = \mathbf{e} \cdot \boldsymbol{b}^{-1} \cdot \mathbf{e}\,. \tag{2.61}$$

Let us denote the eigenvalues, eigenvectors and principal directions of the tensor $\boldsymbol{C}$ by λ_1^2, λ_2^2, λ_3^2; $\mathbf{n}_1^\circ$, $\mathbf{n}_2^\circ$, $\mathbf{n}_3^\circ$ and n_1°, n_2°, n_3° ((n°) designates the coordinate system constituted by the principal directions).

Let us further denote the eigenvalues, eigenvectors and principal directions of the tensor $\boldsymbol{b}^{-1}$ by $1/\lambda_1^2$, $1/\lambda_2^2$, $1/\lambda_3^2$; $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and n_1, n_2, n_3 ((n) designates the coordinate system constituted by the principal directions).

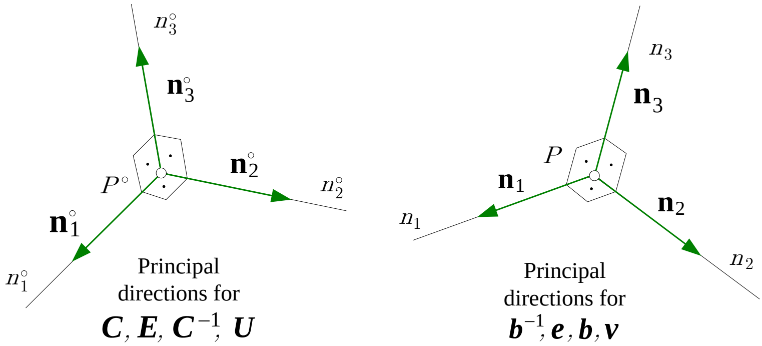

FIGURE 2.5. Principal directions in material and spatial descriptions

Since any direction is principal direction for the unit tensor $\boldsymbol{1}$ it follows form (2.32), (2.48) by taking, in addition to this, into account that the principal directions of a tensor and its inverse are the same that the tensors $\boldsymbol{C}$, $\boldsymbol{E}$, $\boldsymbol{C}^{-1}$ defined in the initial configuration are coaxial. A similar reasoning shows that the tensors $\boldsymbol{b}^{-1}$, $\boldsymbol{e}$, $\boldsymbol{b}$ of the current configuration are also coaxial.

EXERCISE 2.5: Give $\boldsymbol{C}$ and its matrix $\underline{\mathbf{C}}$ as well as $\boldsymbol{E}$ and its matrix $\underline{\mathbf{E}}$ in the coordinate system of the principal axes (n°).

It follows from equations (1.116) and $(2.61)_1$ that

$$\underset{(n^\circ)}{\boldsymbol{C}} = \sum_{A=1}^{3} \lambda_A^2\, \mathbf{n}_A^\circ \circ \mathbf{n}_A^\circ\,, \qquad \underset{(n^\circ)}{\underline{\mathbf{C}}} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}. \tag{2.62}$$

With (2.62) equation (2.32) yields

$$\underset{(n^\circ)}{\boldsymbol{E}} = \frac{1}{2}\big(\underset{(n^\circ)}{\boldsymbol{C}} - \underset{(n^\circ)}{\boldsymbol{1}}\big) = \frac{1}{2}\sum_{A=1}^{3}\left(\lambda_A^2 - 1\right)\,\mathbf{n}_A^\circ \circ \mathbf{n}_A^\circ\,, \qquad \underset{(n^\circ)}{\underline{\mathbf{E}}} = \frac{1}{2}\begin{bmatrix} \lambda_1^2-1 & 0 & 0 \\ 0 & \lambda_2^2-1 & 0 \\ 0 & 0 & \lambda_3^2-1 \end{bmatrix}. \tag{2.63}$$

EXERCISE 2.6: Determine $\boldsymbol{b}^{-1}$ and its matrix $\underline{\mathbf{b}}^{-1}$, $\boldsymbol{b}$ and its matrix $\underline{\mathbf{b}}$ as well as $\boldsymbol{e}$ and its matrix $\underline{\mathbf{e}}$ in the coordinate system of the principal axes (n).

It follows from equations (1.116), (1.121) and $(2.61)_2$ that

$$\underset{(n)}{\boldsymbol{b}^{-1}} = \sum_{\ell=1}^{3} \frac{1}{\lambda_\ell^2}\,\mathbf{n}_\ell \circ \mathbf{n}_\ell\,, \qquad \underset{(n)}{\underline{\mathbf{b}}^{-1}} = \begin{bmatrix} 1/\lambda_1^2 & 0 & 0 \\ 0 & 1/\lambda_2^2 & 0 \\ 0 & 0 & 1/\lambda_3^2 \end{bmatrix}. \tag{2.64}$$

With (2.64) we may write

$$\underset{(n)}{\boldsymbol{b}} = \sum_{\ell=1}^{3} \lambda_\ell^2\,\mathbf{n}_\ell \circ \mathbf{n}_\ell\,, \qquad \underset{(n)}{\underline{\mathbf{b}}} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}. \tag{2.65}$$

Note that $\underset{(n^\circ)}{\underline{\mathbf{C}}} = \underset{(n)}{\underline{\mathbf{b}}}$. (However $\boldsymbol{C} \neq \boldsymbol{b}$.) It also holds that

$$\underset{(n)}{\boldsymbol{e}} = \frac{1}{2}\big(\underset{(n)}{\boldsymbol{1}} - \underset{(n)}{\boldsymbol{b}^{-1}}\big) = \frac{1}{2}\sum_{\ell=1}^{3}\left(1-\frac{1}{\lambda_\ell^2}\right)\mathbf{n}_\ell\circ\mathbf{n}_\ell, \quad \underset{(n)}{\underline{\mathbf{e}}} = \frac{1}{2}\begin{bmatrix} 1-1/\lambda_1^2 & 0 & 0 \\ 0 & 1-1/\lambda_2^2 & 0 \\ 0 & 0 & 1-1/\lambda_3^2 \end{bmatrix}. \tag{2.66}$$

2.2.4. Relations between the Green-Lagrange and Euler-Almansi strain tensors. Let us dot multiply $(2.48)_2$ by $\boldsymbol{F}^T$ from left and $\boldsymbol{F}$ from right. With regard to $(2.48)_1$ and $(2.32)_{1,2}$ we have

$$\boldsymbol{E} = \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F}\,, \qquad E_{AB} = F_{Ak}\,e_{k\ell}\,F_{\ell B}\,. \tag{2.67}$$

We can get a pair of this relationship if we dot multiply it by $\boldsymbol{F}^{-T}$ from left and $\boldsymbol{F}^{-1}$ from right:

$$\boldsymbol{e} = \boldsymbol{F}^{-T} \cdot \boldsymbol{E} \cdot \boldsymbol{F}^{-1}\,, \qquad e_{k\ell} = F^{-1}_{kA}\,E_{AB}\,F^{-1}_{B\ell}\,. \tag{2.68}$$

The above relationships are, in fact, transformations between tensors regarded in material and spatial descriptions. The relationships that are transformations between quantities (vectors or tensors) regarded in material and spatial descriptions are refereed to as a push-forward operation and a pull-back operation. Equation (2.68) shows that the Euler-Almansi strain tensor $\boldsymbol{e}$ (defined in spatial description in the current configuration) is a push-forward of the Green-Lagrange strain tensor $\boldsymbol{E}$ (defined in material description in the initial configuration). The push-back operation is the inverse of the push-forward operation – equation (2.67) shows that the Green-Lagrange strain tensor is a pull-back of the the Euler-Almansi strain tensor.

2.3. The polar decomposition theorem

2.3.1. The polar decomposition theorem and its proof. Results for the finite rotation appear in the polar decomposition theorem which is of fundamental importance in the theory of finite strains. It is worthy of mentioning that the polar decomposition theorem was established by J. Finger [29, 71].

Consider a tensor $\boldsymbol{F}$ which meets the condition

$$\det(\boldsymbol{F}) = |\boldsymbol{F}| = \hat{J} > 0\,. \tag{2.69}$$

We remark that inequality (2.69) is the only precondition for $\boldsymbol{F}$ which, otherwise, can be any tensor of order two, that is, this tensor is not necessarily the same as the deformation gradient.

If precondition (2.69) holds then $\boldsymbol{F}$ can always be given in the form

$$\boxed{\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{R}\,,} \tag{2.70}$$

where $\boldsymbol{U}$, $\boldsymbol{V}$ are positive definite symmetric tensors and $\boldsymbol{R}$ is a rotation tensor.

The tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ are given by

$$\boxed{\boldsymbol{U} = \sqrt{\boldsymbol{F}^T \cdot \boldsymbol{F}}\,; \qquad \boldsymbol{V} = \sqrt{\boldsymbol{F} \cdot \boldsymbol{F}^T}\,.} \tag{2.71}$$

The products $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$ and $\boldsymbol{F} = \boldsymbol{V} \cdot \boldsymbol{R}$ are called right and left polar decompositions of the tensor $\boldsymbol{F}$.

Symmetry. The transformations

$$\left(\boldsymbol{F}^T \cdot \boldsymbol{F}\right)^T = \boldsymbol{F}^T \cdot \left(\boldsymbol{F}^T\right)^T = \boldsymbol{F}^T \cdot \boldsymbol{F}\,,$$
$$\left(\boldsymbol{F} \cdot \boldsymbol{F}^T\right)^T = \left(\boldsymbol{F}^T\right)^T \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{F}^T$$

show that both $\boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{F} \cdot \boldsymbol{F}^T$ are symmetric. Let $\mathbf{v}$ be an arbitrary vector. It also holds that

$$\mathbf{v} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \mathbf{v} = \left(\boldsymbol{F}^T \cdot \mathbf{v}\right) \cdot \left(\boldsymbol{F}^T \cdot \mathbf{v}\right) \geq 0\,,$$
$$\mathbf{v} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \mathbf{v} = (\boldsymbol{F} \cdot \mathbf{v}) \cdot (\boldsymbol{F} \cdot \mathbf{v}) \geq 0\,.$$

Since $|\boldsymbol{F}| > 0$ the tensor $\boldsymbol{F}$ has an inverse. This means that the right sides of equations $\boldsymbol{F} \cdot \mathbf{v} = \mathbf{0}$ and $\boldsymbol{F}^T \cdot \mathbf{v} = \mathbf{0}$ can be zero if and only if $\mathbf{v} = \mathbf{0}$. Consequently,

$$\boldsymbol{F}^T \cdot \boldsymbol{F} \qquad \text{and} \qquad \boldsymbol{F} \cdot \boldsymbol{F}^T$$

are both positive definite tensors.

Uniqueness. Let $\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$ be a right polar decomposition of $\boldsymbol{F}$ in which the tensor $\boldsymbol{R}$ is rotation tensor. Hence the equation

$$\boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{U}^T \cdot \underbrace{\boldsymbol{R}^T \cdot \boldsymbol{R}}_{\boldsymbol{1}} \cdot \boldsymbol{U} = \boldsymbol{U}^2$$

should also be satisfied. Recalling that the square root extraction for a tensor – see definition (1.126) – is a unique operation we can come to the conclusion that there is only one symmetric and positive definite $\boldsymbol{U}$ for which it is true that $\boldsymbol{U}^2 = \boldsymbol{F}^T \cdot \boldsymbol{F}$. Since $\boldsymbol{U}$ is unique so is the rotation tensor

$$\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}\,. \tag{2.72}$$

Existence. Let us define $\boldsymbol{U}$ by equation $(2.71)_1$. Further let

$$\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$$

be the rotation tensor in the left polar decomposition. If $\boldsymbol{R}$ satisfies the relations $\det(\boldsymbol{R}) = 1$ and $\boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{1}$ then it follows that $(2.70)_1$ is really a polar decomposition.

If we take into account the assumption $\det(\boldsymbol{F}) = \hat{J} > 0$ and the fact that $\det(\boldsymbol{U}) = \det(\boldsymbol{F})$ we have

$$\det(\boldsymbol{R}) = \det(\boldsymbol{F})\det(\boldsymbol{U}^{-1}) = \det(\boldsymbol{F})/\det(\boldsymbol{U}) = \hat{J}/\hat{J} = 1\,.$$

On the other hand

$$\boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{U}^{-1} \cdot \underbrace{\boldsymbol{F}^T \cdot \boldsymbol{F}}_{\boldsymbol{U}^2} \cdot \boldsymbol{U}^{-1} = \boldsymbol{1}$$

which shows that $\boldsymbol{R}$ is a rotation tensor. Hence, we have proved the existence and uniqueness of the right polar decomposition.

Left polar decomposition. We shall define $\boldsymbol{V}$ by the following equation

$$\boldsymbol{V} = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T\,. \tag{2.73}$$

Since both $\boldsymbol{R}$ and $\boldsymbol{U}$ are unique so is $\boldsymbol{V}$.

Note that $\boldsymbol{V}$ is symmetric and positive definite.

As regards the symmetry that follows from the transformation

$$\boldsymbol{V}^T = \left(\boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T\right)^T = \boldsymbol{R} \cdot (\boldsymbol{R} \cdot \boldsymbol{U})^T = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T = \boldsymbol{V}\,.$$

Let $\mathbf{v}$ be again an arbitrary vector. Given $\mathbf{v}$ we can write

$$\mathbf{v} \cdot \boldsymbol{V} \cdot \mathbf{v} = \mathbf{v} \cdot \underbrace{\boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^T}_{\boldsymbol{V}} \cdot \mathbf{v} = \mathbf{v} \cdot \boldsymbol{R} \cdot \sqrt{\boldsymbol{U}} \cdot \sqrt{\boldsymbol{U}} \cdot \boldsymbol{R}^T \cdot \mathbf{v} =$$
$$= \left(\sqrt{\boldsymbol{U}} \cdot \boldsymbol{R}^T \cdot \mathbf{v}\right) \cdot \left(\sqrt{\boldsymbol{U}} \cdot \boldsymbol{R}^T \cdot \mathbf{v}\right) = \left(\sqrt{\boldsymbol{U}} \cdot \boldsymbol{R}^T \cdot \mathbf{v}\right)^2 \geq 0\,,$$

in which $\sqrt{\boldsymbol{U}} \cdot \boldsymbol{R}^T$ is invertible since $|\sqrt{\boldsymbol{U}} \cdot \boldsymbol{R}^T| \neq 0$. If this is the case the right side of $\sqrt{\boldsymbol{U}} \cdot \boldsymbol{R}^T \cdot \mathbf{v} = \mathbf{0}$ can be zero if and only if $\mathbf{v} = \mathbf{0}$. The tensor $\boldsymbol{V}$ is, therefore, positive definite.

For the tensor $\boldsymbol{V}$ defined by equation (2.73) it also holds that

$$\boldsymbol{V} \cdot \boldsymbol{R} = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \underbrace{\boldsymbol{R}^T \cdot \boldsymbol{R}}_{\boldsymbol{1}} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{F}\,,$$

which shows that $\boldsymbol{V} \cdot \boldsymbol{R}$ is really the left polar decomposition.

Finally note that

$$\boldsymbol{V}^2 = \underbrace{\boldsymbol{R} \cdot \boldsymbol{U}}_{\boldsymbol{F}} \cdot \underbrace{\boldsymbol{R}^T \cdot \boldsymbol{R}}_{\boldsymbol{1}} \cdot \underbrace{\boldsymbol{U} \cdot \boldsymbol{R}^T}_{\boldsymbol{F}^T} = \boldsymbol{F} \cdot \boldsymbol{F}^T$$

which shows that equation $(2.71)_2$ is also satisfied.

2.3.2. The polar decomposition theorem and the deformation gradient. In accordance with the previous subsection we shall denote the square roots of the positive definite symmetric tensors $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ in the following manner:

$$\left.\begin{aligned} \boldsymbol{U} = \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{F}^T \cdot \boldsymbol{F}}\,, \qquad & \boldsymbol{v} = \sqrt{\boldsymbol{b}} = \sqrt{\boldsymbol{F} \cdot \boldsymbol{F}^T}\,, \\ U_{BS}U_{SA} = C_{BA} = F_{Bk}F_{kA}\,, \qquad & v_{ks}v_{s\ell} = b_{k\ell} = F_{kB}F_{B\ell}\,. \end{aligned}\right\} \tag{2.74}$$

With regard to (2.62) and (2.65) we can give $\boldsymbol{U}$ and $\boldsymbol{v}$ in the coordinate systems of the principal axes:

$$\underset{(n^\circ)}{\boldsymbol{U}} = \sqrt{\boldsymbol{C}} = \sum_{A=1}^{3} \lambda_A \, \mathbf{n}_A^\circ \circ \mathbf{n}_A^\circ \, , \qquad \underset{(n^\circ)}{\underline{\mathbf{U}}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} , \tag{2.75a}$$

$$\underset{(n)}{\boldsymbol{v}} = \sqrt{\boldsymbol{b}} = \sum_{a=1}^{3} \lambda_a \, \mathbf{n}_a \circ \mathbf{n}_a \, , \qquad \underset{(n)}{\underline{\boldsymbol{v}}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} , \tag{2.75b}$$

$$\lambda_A = \lambda_a \, . \tag{2.75c}$$

With $\boldsymbol{U}$ and $\boldsymbol{v}$ the polar decompositions of the deformation gradient $\boldsymbol{F}$ is of the form

$$\left. \begin{aligned} \boldsymbol{F} &= \boldsymbol{R} \cdot \boldsymbol{U} \, , & \boldsymbol{F} &= \boldsymbol{v} \cdot \boldsymbol{R} \, , \\ F_{kA} &= R_{kB} U_{BA} \, , & F_{kA} &= v_{k\ell} R_{\ell A} \, . \end{aligned} \right\} \tag{2.76}$$

The tensor [$\boldsymbol{U}$]{$\boldsymbol{v}$} is called [right]{left} stretch tensor. It is obvious that $\boldsymbol{U}$ is coaxial with $\boldsymbol{C}$ while $\boldsymbol{v}$ is coaxial with $\boldsymbol{b}^{-1}$ – see Figure 2.5.

Exercise 2.7: Clarify how the principal directions $\mathbf{n}_A^\circ$ and $\mathbf{n}_a$ are related to each other.
It follows from $(2.76)_1$ that

$$\boldsymbol{F} \cdot \mathbf{n}_A^\circ = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \mathbf{n}_A^\circ = \boldsymbol{v} \cdot \boldsymbol{R} \cdot \mathbf{n}_A^\circ \tag{2.77}$$

We can also write by utilizing equation (2.75a) that

$$\boldsymbol{U} \cdot \mathbf{n}_A^\circ = \lambda_A \, \mathbf{n}_A^\circ \qquad \text{(no sum on } A) \tag{2.78}$$

Let us now substitute (2.78) into (2.77). We have

$$\boldsymbol{R} \cdot \underbrace{\boldsymbol{U} \cdot \mathbf{n}_A^\circ}_{\lambda_A \, \mathbf{n}_A^\circ} = \boldsymbol{v} \cdot \boldsymbol{R} \cdot \mathbf{n}_A^\circ \qquad \text{(no sum on } A)$$

or

$$\boldsymbol{v} \cdot \boxed{\boldsymbol{R} \cdot \mathbf{n}_A^\circ} = \lambda_A \boxed{\boldsymbol{R} \cdot \mathbf{n}_A^\circ} \qquad \text{(no sum on } A)$$

On the other hand it holds

$$\boldsymbol{v} \cdot \boxed{\mathbf{n}_a} = \lambda_a \boxed{\mathbf{n}_a} \, , \qquad \lambda_A = \lambda_a \, , \qquad \text{(no sum on } a = A)$$

Hence

$$\mathbf{n}_a = \boldsymbol{R} \cdot \mathbf{n}_A^\circ \, . \tag{2.79}$$

In words: The rotation tensor $\boldsymbol{R}$ as a two point tensor shifts the unit vector $\mathbf{n}_A^\circ$ from P° to P and then rotates it into the unit vector $\mathbf{n}_a$.

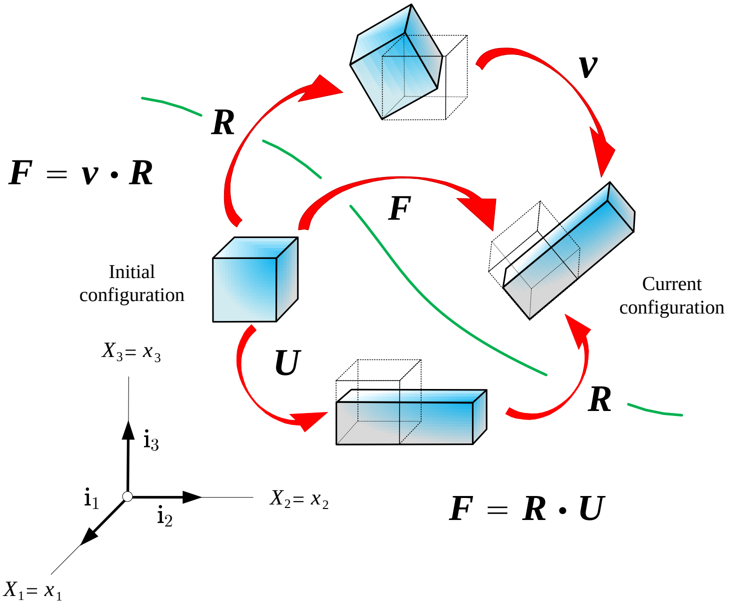

FIGURE 2.6. The geometry of the finite uniaxial deformations

Figure[7] 2.6 is a geometrical representation of the mapping $\mathrm{d}\mathbf{x} = \boldsymbol{F} \cdot \mathrm{d}\mathbf{X}$ under the assumption of a uniaxial deformation. The physical content of the mapping is explained below via equations (2.80a) and (2.80b):

$$\mathrm{d}\mathbf{x} = \boldsymbol{F} \cdot \mathrm{d}\mathbf{X} = \boldsymbol{R} \cdot \underbrace{\underbrace{\boldsymbol{U} \cdot \mathrm{d}\mathbf{X}}_{\text{Pure deformation of the cube into a rectangular parallelepiped – this preceeds the rigid body motion.}}}_{\text{Rigid body motion of the deformed cube: shift followed by a rotation.}} \tag{2.80a}$$

and

$$\mathrm{d}\mathbf{x} = \boldsymbol{F} \cdot \mathrm{d}\mathbf{X} = \boldsymbol{v} \cdot \underbrace{\underbrace{\boldsymbol{R} \cdot \mathrm{d}\mathbf{X}}_{\text{Rigid body motion of the undeformed cube: shift followed by a rotation.}}}_{\text{Pure deformation of the cube into a rectangular parallelepiped in the current configuration.}} \,. \tag{2.80b}$$

[7]Figure 2.6 is taken form the url address https://en.wikipedia.org/wiki/Finite_strain_theory

2.4. Generalization of the strain tensor concept

2.4.1. Strain tensors in the initial configuration. Equations

$$\begin{aligned} \boldsymbol{E}^{(0)} &= \ln \boldsymbol{U}\,, \\ \boldsymbol{E}^{(1)} &= \boldsymbol{H} = \boldsymbol{U} - \boldsymbol{1}\,, \\ \boldsymbol{E}^{(2)} &= \boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{1}\right) \end{aligned} \tag{2.81}$$

define the Hencky[8] strain tensor $\boldsymbol{E}^{(0)}$ [40], the Biot strain tensor $\boldsymbol{E}^{(1)} = \boldsymbol{H}$ [50] and the Green-Lagrange strain tensor $\boldsymbol{E}^{(2)} = \boldsymbol{E}$.

2.4.2. Strain tensors in the current configuration. Similar relationships, i.e., the equations

$$\begin{aligned} \boldsymbol{e}^{(0)} &= \ln \boldsymbol{v}\,, \\ \boldsymbol{e}^{(1)} &= \boldsymbol{h} = \boldsymbol{1} - \boldsymbol{v}^{-1}\,, \\ \boldsymbol{e}^{(2)} &= \boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{1} - \boldsymbol{v}^{-2}\right) \end{aligned} \tag{2.82}$$

define the spatial Hencky strain tensor $\boldsymbol{e}^{(0)}$, the spatial Biot strain tensor $\boldsymbol{e}^{(1)} = \boldsymbol{h}$ and the Euler-Almansi strain tensor $\boldsymbol{e}^{(2)} = \boldsymbol{e}$.

Note that the strain tensors $[\boldsymbol{E}^{(0)}, \boldsymbol{E}^{(1)} = \boldsymbol{H}$ and $\boldsymbol{E}^{(2)} = \boldsymbol{E}]$ $\{\boldsymbol{e}^{(0)}, \boldsymbol{e}^{(1)} = \boldsymbol{h}$ and $\boldsymbol{e}^{(2)} = \boldsymbol{e}\}$ are coaxial.

Remark 2.18: Equations (2.81) and (2.82) are the special cases of the equations

$$\boldsymbol{E}^{(n)} = \begin{cases} \ln \boldsymbol{U} & \text{if} \quad n = 0 \\ \dfrac{1}{n}\left(\boldsymbol{U}^n - \boldsymbol{1}\right) & \text{if} \quad n > 0 \end{cases} \tag{2.83}$$

and

$$\boldsymbol{e}^{(n)} = \begin{cases} \ln \boldsymbol{v} & \text{if} \quad n = 0 \\ \dfrac{1}{n}\left(\boldsymbol{1} - \boldsymbol{v}^{-n}\right) & \text{if} \quad n > 0 \end{cases} \tag{2.84}$$

where n is a non negative integer.

2.5. Further strain measures

2.5.1. Nanson's formula. Figure 2.7 shows the differential (infinitesimal) material surface elements

$$\begin{aligned} \mathrm{d}\mathbf{A}^\circ &= \mathbf{n}^\circ \mathrm{d}A^\circ = \mathrm{d}\mathbf{X}_I \times \mathrm{d}\mathbf{X}_{II}\,, & \mathrm{d}\mathbf{A} &= \mathbf{n}\,\mathrm{d}A = \mathrm{d}\mathbf{x}_I \times \mathrm{d}\mathbf{x}_{II}\,, \\ |\mathbf{n}^\circ| &= 1\,, & |\mathbf{n}| &= 1 \end{aligned} \tag{2.85}$$

given in terms of $\mathrm{d}\mathbf{X}_I$, $\mathrm{d}\mathbf{X}_{II}$ in the initial configuration and in terms of $\mathrm{d}\mathbf{x}_I$, $\mathrm{d}\mathbf{x}_{II}$ in the current configuration. Using indicial notation we can write

$$\begin{aligned} \mathrm{d}A^\circ_P &= \mathrm{d}A^\circ n^\circ_P = e_{PQR}\,\mathrm{d}X^I_Q\,\mathrm{d}X^{II}_R\,, \\ \mathrm{d}A_k &= \mathrm{d}A n_k = e_{k\ell m}\,\mathrm{d}x^I_\ell\,\mathrm{d}x^{II}_m\,. \end{aligned} \tag{2.86}$$

[8] Heinrich Hencky, 1885-1951

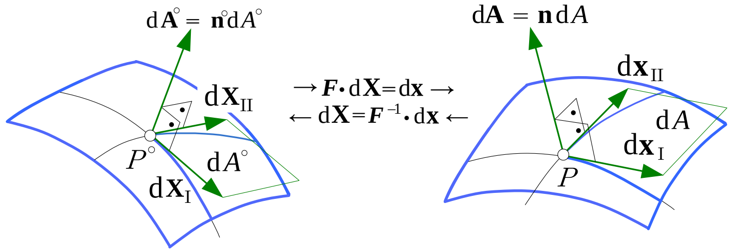

FIGURE 2.7. Deformation of the surface element $\mathrm{d}A^\circ$ into the surface element $\mathrm{d}A$

Making use of equation (2.14)$_2$ we can rewrite (2.86)$_2$:

$$\mathrm{d}A_k = \mathrm{d}A n_k = e_{k\ell m}\, F_{\ell Q} F_{mR}\, \mathrm{d}X_Q^I\, \mathrm{d}X_R^{II}\,.$$

If we now multiply throughout by F_{kP} we get

$$F_{kP} n_k \mathrm{d}A = e_{k\ell m}\, F_{kP} F_{\ell Q} F_{mR}\, \mathrm{d}X_Q^I\, \mathrm{d}X_R^{II}\,. \tag{2.87}$$

By utilizing (1.47) and (2.20) we may verify that

$$e_{k\ell m}\, F_{kP} F_{\ell Q} F_{mR} = e_{PQR}\det(F_{aB}) = e_{PQR} J\,. \tag{2.88}$$

Substituting this result into (2.87) and taking then (2.86)$_1$ into account yields

$$n_k F_{kP} \mathrm{d}A = J\, e_{PQR}\, \mathrm{d}X_Q^I\, \mathrm{d}X_R^{II} = J\, \mathrm{d}A_P^\circ\,.$$

The final result can be obtained if we multiply the above equation by F_{Ps}^{-1}. We arrive at the following equation

$$n_k \underbrace{F_{kP} F_{Ps}^{-1}}_{\delta_{ks}} \mathrm{d}A = n_s \mathrm{d}A = \mathrm{d}A_s = J\, F_{Ps}^{-1} \mathrm{d}A_P^\circ = J\, F_{sP}^{-1} \mathrm{d}A_P^\circ = J\, F_{sP}^{-1} n_P^\circ\, \mathrm{d}A^\circ$$

or

$$\boxed{\mathrm{d}A_s = J\, F_{sP}^{-1}\, \mathrm{d}A_P^\circ = J\, F_{sP}^{-1} n_P^\circ\, \mathrm{d}A^\circ\,.} \tag{2.89a}$$

We can give this equation in direct notation as well:

$$\boxed{\mathrm{d}\mathbf{A} = J\, \boldsymbol{F}^{-T} \cdot \mathrm{d}\mathbf{A}^\circ = J\, \boldsymbol{F}^{-T} \cdot \mathbf{n}^\circ\, \mathrm{d}A^\circ\,.} \tag{2.89b}$$

Equation (2.89) is known as Nanson's[9] formula [21]. By using Nanson's formula and relation (2.85) between the vectorial and scalar surface elements we can write

$$(\mathrm{d}A)^2 = \mathrm{d}\mathbf{A}\cdot\mathrm{d}\mathbf{A} = J^2\, \mathrm{d}\mathbf{A}^\circ\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}\cdot\mathrm{d}\mathbf{A}^\circ = J^2\, \mathbf{n}^\circ\cdot\underbrace{\boldsymbol{F}^{-1}\cdot\,\boldsymbol{F}^{-T}}_{\boldsymbol{C}^{-1}}\cdot\mathbf{n}^\circ\, (\mathrm{d}A^\circ)^2 \tag{2.90a}$$

or

$$(\mathrm{d}A^\circ)^2 = \mathrm{d}\mathbf{A}^\circ\cdot\mathrm{d}\mathbf{A}^\circ = \frac{1}{J^2}\, \mathrm{d}\mathbf{A}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T\cdot\mathrm{d}\mathbf{A} = \frac{1}{J^2}\, \mathbf{n}\cdot\underbrace{\boldsymbol{F}\cdot\boldsymbol{F}^T}_{\boldsymbol{b}}\cdot\mathbf{n}\, (\mathrm{d}A)^2\,. \tag{2.90b}$$

[9]Edward J. Nanson, 1850-1936

Thus

$$\boxed{\mathrm{d}A = J\sqrt{\mathbf{n}^\circ \cdot \boldsymbol{C}^{-1} \cdot \mathbf{n}^\circ}\,\mathrm{d}A^\circ \qquad \text{and} \qquad \mathrm{d}A = \frac{1}{J}\frac{\mathrm{d}A^\circ}{\sqrt{\mathbf{n}\cdot\boldsymbol{b}\cdot\mathbf{n}}}\,.} \tag{2.91}$$

A comparison of these two equations yields the definition of the area element ratio:

$$\boxed{\lambda_A = \frac{\mathrm{d}A}{\mathrm{d}A^\circ} = J\sqrt{\mathbf{n}^\circ \cdot \boldsymbol{C}^{-1} \cdot \mathbf{n}^\circ} = \frac{1}{J}\frac{1}{\sqrt{\mathbf{n}\cdot\boldsymbol{b}\cdot\mathbf{n}}}\,.} \tag{2.92}$$

2.5.2. Volume change. It is clear from Figure 2.8 that the infinitesimal volume element in the initial configuration is given by the following equation:

$$\mathrm{d}V^\circ = [\mathrm{d}\mathbf{X}_I\,\mathrm{d}\mathbf{X}_{II}\,\mathrm{d}\mathbf{X}_{III}] = e_{IJK}\,\mathrm{d}X_I^I\,\mathrm{d}X_J^{II}\,\mathrm{d}X_K^{III}\,. \tag{2.93a}$$

The infinitesimal volume element in the current configuration

$$\mathrm{d}V = [\mathrm{d}\mathbf{x}_I\,\mathrm{d}\mathbf{x}_{II}\,\mathrm{d}\mathbf{x}_{III}] = e_{pqr}\mathrm{d}x_p^I\,\mathrm{d}x_q^{II}\,\mathrm{d}x_r^{III} \tag{2.93b}$$

can be related to the infinitesimal volume element in the initial configuration if we substitute $(2.14)_2$ and then apply the relation $(2.88)_2$. We get

$$\mathrm{d}V = \underbrace{e_{pqr}F_{pI}F_{qJ}F_{rK}}_{e_{IJK}J}\,\mathrm{d}X_I^I\,\mathrm{d}X_J^{II}\,\mathrm{d}X_K^{III} = \underset{(2.93)}{\uparrow} = J\,\underbrace{e_{IJK}\mathrm{d}X_I^I\,\mathrm{d}X_J^{II}\,\mathrm{d}X_K^{III}}_{\mathrm{d}V^\circ}$$

or

$$\boxed{\mathrm{d}V = J\,\mathrm{d}V^\circ\,, \qquad \lambda_V = \frac{\mathrm{d}V}{\mathrm{d}V^\circ} = J\,,} \tag{2.94}$$

where λ_V is the volume element ratio.

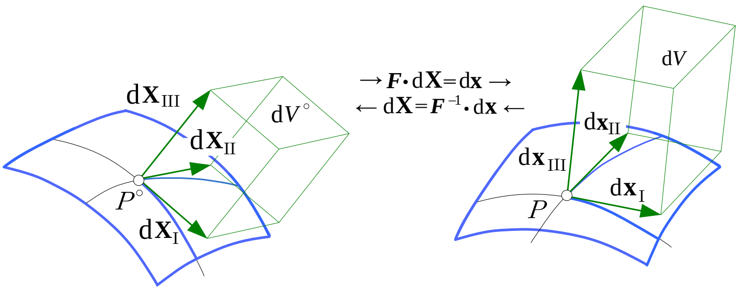

FIGURE 2.8. Deformation of the volume element $\mathrm{d}V^\circ$ into the volume element $\mathrm{d}V$

If $J = 1$ there is no local volume change, i.e., $\mathrm{d}V = \mathrm{d}V^\circ$. The deformation for which $J = 1$ everywhere within the body is called volume preserving or isochoric deformation.

2.6. Multiplicative decomposition

The deformation gradient can be multiplicatively decomposed into two parts:

$$\boxed{\boldsymbol{F} = \underbrace{J^{1/3}\boldsymbol{1}}_{\boldsymbol{F}_{\text{vpr}}} \cdot \underbrace{\boldsymbol{F}J^{-1/3}}_{\boldsymbol{F}_{\text{tor}}},} \tag{2.95}$$

where

$$|\underbrace{J^{1/3}\boldsymbol{1}}_{\boldsymbol{F}_{\text{vpr}}}| = \det(\underbrace{J^{1/3}\boldsymbol{1}}_{\boldsymbol{F}_{\text{vpr}}}) = \begin{vmatrix} J^{1/3} & 0 & 0 \\ 0 & J^{1/3} & 0 \\ 0 & 0 & J^{1/3} \end{vmatrix} = J \tag{2.96a}$$

and

$$|\boldsymbol{F}J^{-1/3}| = \det(\boldsymbol{F}J^{-1/3}) = \det(\boldsymbol{F})(J^{-1/3})^3 = 1. \tag{2.96b}$$

The determinant of $\boldsymbol{F}_{\text{vpr}}$ is equal to J, therefore, the tensor

$$\boxed{\boldsymbol{F}_{\text{vpr}} = J^{1/3}\boldsymbol{1}} \tag{2.97a}$$

is the volume preserving part of the deformation gradient. In contrast to this the determinant of $\boldsymbol{F}_{\text{tor}}$ is equal to 1, consequently, the tensor

$$\boxed{\boldsymbol{F}_{\text{tor}} = \boldsymbol{F}J^{-1/3}} \tag{2.97b}$$

is the isochoric or distortional part of the deformation gradient.

2.7. Compatibility conditions

The compatibility conditions constitute a special problem in the continuum mechanics of solid bodies.

With the 3 displacement components of the displacement field $\mathbf{u}^\circ$ we can determine the 6 independent components of the Green-Lagrange strain tensor $\boldsymbol{E}$ by using equation (2.39).

In the opposite case when we know the 6 components of the Green-Lagrange strain tensor and would like to determine the 3 components of the displacement field $\mathbf{u}^\circ$ we have to solve the 6 partial differential equations (2.39) for finding 3 unknowns. This problem is, therefore, overdetermined (we have more equations than there are unknowns) and the solution exits only if the tensor field $\boldsymbol{E}$ satisfies some restrictive conditions. These conditions are called compatibility conditions.

In the nonlinear theory of deformations it is very difficult (in fact it is a hopeless problem) to eliminate the displacement components from the kinematic equations (2.39). This is the reason why the Riemann theory has come into general use. The essence of this theory is as follows.

It is a fundamental assumption that the geometrical space is euclidean both in the initial configuration and in the final one. Consequently, the metric tensor of the deformed continuum should be a positive definite tensor (each metric tensor satisfies this requirement) and, in addition to this, should also satisfy the fourth order Riemann-Christoffel curvature tensor.

2.8. Problems

PROBLEM 2.1: Let

$$x_1 = X_2 + X_3 - X_1\,, \qquad x_2 = X_2 + X_3\,, \qquad x_3 = X_2 - 2X_3$$

be the motion law in material description. Determine $\boldsymbol{F}$, $\boldsymbol{F}^{-1}$ and J.

PROBLEM 2.2: Let $\boldsymbol{\chi} = (X_1 + a\,t\,X_2)\,\mathbf{i}_1 + (X_2 - a\,t\,X_1)\,\mathbf{i}_2 + X_3\mathbf{i}_3$ be the motion law in material description where a is a constant. Find (a) the deformation gradient, (b) the inverse motion law and (c) the inverse deformation gradient.

PROBLEM 2.3: Given the temperature distribution in terms of the material coordinates X_A within the cube of Exercise 2.1: $\theta = B\,(1 + X_1X_2)$ where B is a constant. Find the spatial description of the temperature field.

PROBLEM 2.4: Find the displacement field in material and spatial descriptions within the cube of Exercise 2.1.

PROBLEM 2.5: Making use of the motion law of Exercise 2.1 determine, within the cube, (a) the deformation gradient in material and spatial descriptions and (b) prove that

$$\underset{(3\times 3)}{\underline{\mathbf{F}}}^{\,-1} = \left[F^{-1}_{B\ell}\right] = \begin{bmatrix} 1 & -2a_1\,(x_2 - a_2) & 0 \\ 0 & 1 & 0 \\ 0 & -a_3x_3/\left[1+a_3\,(x_2-a_2)\right]^2 & 1/\left[1+a_3\,(x_2-a_2)\right] \end{bmatrix} \qquad (2.98)$$

is the inverse deformation gradient in spatial description.

PROBLEM 2.6: Find the deformation gradient in cylindrical coordinate systems.

PROBLEM 2.7: Assume that $r = R$, $\vartheta = \Theta + z\hat{\vartheta}$ and $z = Z$ in the previous Problem. This is the case for a rod with circular cross section subjected to twisting – it is assumed that the cross sections rotate in their own plane. Then $\hat{\vartheta}$ is the angle of rotation for a unit length. Find the displacement gradient utilizing the notations given in Figure 4.1.

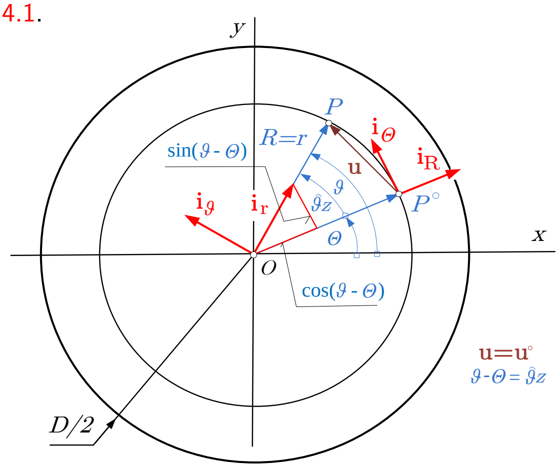

FIGURE 2.9. Cross section of the rod considerred

Problem 2.8: Show that the Green-Lagrange strain tensor is of the form

$$\underset{(3\times3)}{\underline{\mathbf{E}}} = [E_{AB}] = \begin{bmatrix} 0 & a_1\,X_2 & 0 \\ a_1\,X_2 & 2a_1^2X_2^2 + \frac{1}{2}a_3^2X_3^2 & \frac{1}{2}a_3X_3\,(1 + a_3X_2) \\ 0 & \frac{1}{2}a_3X_3\,(1 + a_3X_2) & a_3X_2\left(1 + \frac{1}{2}a_3X_2\right) \end{bmatrix} \tag{2.99}$$

for the motion given in Exercise 2.1.

Problem 2.9: For the deformation

$$x_1 = X_1\,, \qquad x_2 = -3X_3\,, \qquad x_3 = 2X_2$$

find $\boldsymbol{F}$, $\boldsymbol{U}$, $\boldsymbol{v}$ and $\boldsymbol{R}$.

Problem 2.10: For the deformation

$$x_1 = 2X_1 - 2X_2\,, \qquad x_2 = X_1 + X_2, \qquad x_3 = X_3$$

find $\boldsymbol{F}$, $\boldsymbol{U}$, $\boldsymbol{v}$ and $\boldsymbol{R}$. Prove that the matrix of the left stretch tensor $\boldsymbol{v}$ is a diagonal matrix.

Problem 2.11: For the deformation

$$x_1 = 2X_3\,, \qquad x_2 = -X_1\,, \qquad x_3 = -2X_2 + 3X_3$$

determine $\boldsymbol{F}$, $\boldsymbol{U}$, $\boldsymbol{v}$ and $\boldsymbol{R}$.

Problem 2.12: Given the deformation in the following form:

$$x_1 = \sqrt{2X_1}\cos X_2\,, \qquad x_2 = \sqrt{2X_1}\sin X_2\,, \qquad x_3 = X_3$$

Find the inverse motion law, the deformation gradients $\boldsymbol{F}$, $\boldsymbol{F}^{-1}$ and show that the above deformation is volume preserving.

Problem 2.13: For the deformation considered in Problem **2.10** find the right Cauchy-Green tensor, the Green-Lagrange strain tensor, the principal directions and stretches in the initial configuration, the right stretch tensor, the rotation tensor and the principal directions in the current configuration.

Problem 2.14: Show that the Green Lagrange strain tensor and the Euler Almansi strain tensor are independent of the rigid body rotation (of the tensor $\boldsymbol{R}$). (Hint: Make use of the polar decomposition theorem.)

Problem 2.15: Assume that the displacement field is given by the following equations:

$$u_1 = a(2X_1^2 + X_1X_2)\,, \qquad u_2 = aX_2^2\,, \qquad u_3 = 0; \qquad a = 10^{-4}.$$

Find the axial strains in the directions $\mathbf{i}_1$ and $\mathbf{i}_2$ at the point $P(1,1,0)$. What is the angle change between these directions?

CHAPTER 3

Time derivatives

3.1. Velocity and acceleration

A material point moves along a path determined by the motion law of the continuum. Its velocity is, therefore, given by the equation

$$\boxed{\mathbf{v} = \mathbf{v}(X_1, X_2, X_3; t) = \frac{\mathrm{d}\boldsymbol{\chi}(X_1, X_2, X_3; t)}{\mathrm{d}t} = \left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_{(X)} = \frac{\partial \mathbf{x}}{\partial t}\,,} \tag{3.1}$$

where the subscript (X) denotes that the time derivative is taken for the material point with coordinates X_ℓ. If we take into account that the position vector $\mathbf{X} = X_A \mathbf{i}_A$ in resolution (2.10) is independent of time we get the velocity field:

$$\begin{aligned}\mathbf{v} = v_\ell \mathbf{i}_\ell = \frac{\partial}{\partial t}(\chi_\ell \mathbf{i}_\ell) = \frac{\partial}{\partial t}(x_\ell \mathbf{i}_\ell) = \\ = \frac{\partial}{\partial t}\left[(X_1 + u_1^\circ)\,\mathbf{i}_1 + (X_2 + u_2^\circ)\,\mathbf{i}_2 + (X_3 + u_3^\circ)\,\mathbf{i}_3\right] = \\ = v_1^\circ \mathbf{i}_1 + v_2^\circ \mathbf{i}_2 + v_3^\circ \mathbf{i}_3 = \mathbf{v}^\circ\,,\end{aligned} \tag{3.2a}$$

where

$$\begin{gathered} v_1^\circ(X_1, X_2, X_3; t) = \frac{\partial \chi_1}{\partial t} = \frac{\partial u_1^\circ}{\partial t}\,, \quad v_2^\circ(X_1, X_2, X_3; t) = \frac{\partial \chi_2}{\partial t} = \frac{\partial u_2^\circ}{\partial t}\,, \\ v_3^\circ(X_1, X_2, X_3; t) = \frac{\partial \chi_3}{\partial t} = \frac{\partial u_3^\circ}{\partial t}\,. \end{gathered} \tag{3.2b}$$

are the three velocity components.
With the velocity

$$\boxed{\mathbf{a} = \mathbf{a}(X_1, X_2, X_3; t) = \frac{\mathrm{d}\mathbf{v}(X_1, X_2, X_3; t)}{\mathrm{d}t} = \left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_{(X)} = \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial^2 \boldsymbol{\chi}}{\partial t^2} = \frac{\partial^2 \mathbf{x}}{\partial t^2}} \tag{3.3a}$$

is the acceleration field. On the basis of what has been said above it is not too difficult to check that

$$\begin{gathered} a_1^\circ(X_1, X_2, X_3; t) = \frac{\partial v_1}{\partial t} = \frac{\partial^2 \chi_1}{\partial t^2}\,, \quad a_2^\circ(X_1, X_2, X_3; t) = \frac{\partial v_2}{\partial t} = \frac{\partial^2 \chi_2}{\partial t^2}\,, \\ a_3^\circ(X_1, X_2, X_3; t) = \frac{\partial v_3}{\partial t} = \frac{\partial^2 \chi_3}{\partial t^2} \end{gathered} \tag{3.3b}$$

are the three acceleration components.

For a real motion of continuum mapping $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is one to one, i.e., there exists the inverse motion law $\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x})$. Consequently, if we substitute the components of the inverse motion law $X_1 = \chi_1^{-1}(x_1, x_2, x_3; t)$, $X_2 = \chi_2^{-1}(x_1, x_2, x_3; t)$, $X_3 = \chi_3^{-1}(x_1, x_2, x_3; t)$ into equations (3.2b) and (3.3b) we get velocity and acceleration components at time t in spatial description:

$$v_1 = v_1(x_1, x_2, x_3; t) \; , \quad v_2 = v_2(x_1, x_2, x_3; t) \; , \quad v_3 = v_3(x_1, x_2, x_3; t)$$
$$\mathbf{v} = \mathbf{v}(x_1, x_2, x_3; t) = v_1(x_1, x_2, x_3; t)\,\mathbf{i}_1 + v_2(x_1, x_2, x_3; t)\,\mathbf{i}_2 + v_3(x_1, x_2, x_3; t)\,\mathbf{i}_3 \tag{3.4}$$

and

$$a_1 = a_1(x_1, x_2, x_3; t) \; , \quad a_2 = a_2(x_1, x_2, x_3; t) \; , \quad a_3 = a_3(x_1, x_2, x_3; t)$$
$$\mathbf{a} = \mathbf{a}(x_1, x_2, x_3; t) = a_1(x_1, x_2, x_3; t)\,\mathbf{i}_1 + a_2(x_1, x_2, x_3; t)\,\mathbf{i}_2 + a_3(x_1, x_2, x_3; t)\,\mathbf{i}_3 \, . \tag{3.5}$$

Remark 3.1: Let us assume that we know the velocity field in spatial description. In this case we may also determine the acceleration field in spatial description. The way how to do it is considered in Subsection 3.8 which is devoted to the concept of material time derivatives.

Exercise 3.1: Assume that

$$x_1 = X_1 e^t - X_2 e^{-t} \; , \quad x_2 = X_1 e^t + X_2 e^{-t} \; , \quad x_3 = X_3$$

is the motion law. Determine the velocity and acceleration fields both in material [$\mathbf{v} = \mathbf{v}^\circ(\mathbf{X}; t)$; $\mathbf{a} = \mathbf{a}^\circ(\mathbf{X}; t)$] and in spatial descriptions [$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$; $\mathbf{a} = \mathbf{a}(\mathbf{x}, t)$]. Calculating time derivatives we have

$$v_1^\circ = X_1 e^t + X_2 e^{-t} \; , \quad v_2^\circ = X_1 e^t - X_2 e^{-t} \; , \quad v_3^\circ = 0 \; ,$$
$$a_1^\circ = X_1 e^t - X_2 e^{-t} \; , \quad a_2^\circ = X_1 e^t + X_2 e^{-t} \; , \quad a_3^\circ = 0 \; .$$

In spatial description

$$v_1 = x_2 \; , \qquad v_2 = x_1 \; , \quad v_3 = 0 \; ,$$
$$a_1 = x_1 \; , \qquad a_2 = x_2 \; , \quad a_3 = 0 \; .$$

is the result.

3.2. Velocity gradient

If we linearize the velocity field in the neighborhood of the spatial point P we get:

$$\mathrm{d}\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_1} \underbrace{\mathbf{i}_1 \cdot \mathrm{d}\mathbf{x}}_{\mathrm{d}x_1} + \frac{\partial \mathbf{v}}{\partial x_2} \underbrace{\mathbf{i}_2 \cdot \mathrm{d}\mathbf{x}}_{\mathrm{d}x_2} + \frac{\partial \mathbf{v}}{\partial x_3} \underbrace{\mathbf{i}_3 \cdot \mathrm{d}\mathbf{x}}_{\mathrm{d}x_3} =$$
$$= \underbrace{\left(\frac{\partial \mathbf{v}}{\partial x_1} \circ \mathbf{i}_1 + \frac{\partial \mathbf{v}}{\partial x_2} \circ \mathbf{i}_2 + \frac{\partial \mathbf{v}}{\partial x_3} \circ \mathbf{i}_3 \right)}_{\boldsymbol{l}} \cdot \mathrm{d}\mathbf{x}$$

or

$$\boxed{\mathrm{d}\mathbf{v} = \boldsymbol{l} \cdot \mathrm{d}\mathbf{x}\,, \qquad \mathrm{d}v_p = l_{pq}\,\mathrm{d}x_q\,,} \tag{3.6a}$$

where

$$\boxed{\boldsymbol{l} = \mathbf{v} \circ \nabla = \frac{\partial \mathbf{v}}{\partial x_1} \circ \mathbf{i}_1 + \frac{\partial \mathbf{v}}{\partial x_2} \circ \mathbf{i}_2 + \frac{\partial \mathbf{v}}{\partial x_3} \circ \mathbf{i}_3\,, \qquad l_{pq} = \frac{\partial v_p}{\partial x_q} = v_{p,q}} \tag{3.6b}$$

is the spatial velocity gradient. Using equation (3.6b) we can give its matrix in the following form:

$$\underline{\boldsymbol{l}} = \left[\ \frac{\partial \underline{\mathbf{v}}}{\partial x_1}\ \Bigg|\ \frac{\partial \underline{\mathbf{v}}}{\partial x_2}\ \Bigg|\ \frac{\partial \underline{\mathbf{v}}}{\partial x_3}\ \right] = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_1}{\partial x_2} & \dfrac{\partial v_1}{\partial x_3} \\ \dfrac{\partial v_2}{\partial x_1} & \dfrac{\partial v_2}{\partial x_2} & \dfrac{\partial v_2}{\partial x_3} \\ \dfrac{\partial v_3}{\partial x_1} & \dfrac{\partial v_3}{\partial x_2} & \dfrac{\partial v_3}{\partial x_3} \end{bmatrix}. \tag{3.7}$$

EXERCISE 3.2: Determine the velocity field and its gradient in the cylindrical coordinate system $(R\vartheta z)$.
In this coordinate system

$$\mathbf{x} = R\mathbf{i}_R + z\mathbf{i}_z \tag{3.8}$$

is the position vector of the material point P in the current configuration (see Figure 1.13 for the details), where $R = \chi_R\,(R^\circ, \vartheta^\circ, z^\circ; t)$, $\vartheta = \chi_\vartheta\,(R^\circ, \vartheta^\circ, z^\circ; t)$ and $z = \chi_z\,(R^\circ, \vartheta^\circ, z^\circ; t)$. Deriving equation (3.8) with respect to time and taking relationships (1.186b) into account we have

$$\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t} = \mathbf{i}_R \underbrace{\frac{\partial \chi_R}{\partial t}}_{v_R} + \mathbf{i}_\vartheta \underbrace{R\frac{\partial \chi_\vartheta}{\partial t}}_{v_\vartheta} + \mathbf{i}_z \underbrace{\frac{\partial \chi_z}{\partial t}}_{v_z} = v_R\,\mathbf{i}_R + v_\vartheta\,\mathbf{i}_\vartheta + v_z\,\mathbf{e}_z\,. \tag{3.9}$$

With regard to equations (1.198) and (1.199) we can give the velocity gradient and its matrix as well:

$$\boldsymbol{l} = \mathbf{v} \circ \nabla = \underbrace{\frac{\partial \mathbf{v}}{\partial R}}_{\boldsymbol{l}_R} \circ \mathbf{i}_R + \underbrace{\frac{1}{R}\frac{\partial \mathbf{v}}{\partial \vartheta}}_{\boldsymbol{l}_\vartheta} \circ \mathbf{i}_\vartheta + \underbrace{\frac{\partial \mathbf{v}}{\partial z}}_{\boldsymbol{l}_z} \circ \mathbf{i}_z\,, \tag{3.10a}$$

$$\underset{(R\vartheta z)}{\underline{\boldsymbol{l}}} = \left[\ \underset{(R\vartheta z)}{\underline{\boldsymbol{l}_R}}\ \Bigg|\ \underset{(R\vartheta z)}{\underline{\boldsymbol{l}_\vartheta}}\ \Bigg|\ \underset{(R\vartheta z)}{\underline{\boldsymbol{l}_z}}\ \right] = \begin{bmatrix} l_{RR} & l_{R\vartheta} & l_{Rz} \\ l_{\vartheta R} & l_{\vartheta\vartheta} & l_{\vartheta z} \\ l_{zR} & l_{z\vartheta} & l_{zz} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial v_R}{\partial R} & \dfrac{1}{R}\left(\dfrac{\partial v_R}{\partial \vartheta} - v_\vartheta\right) & \dfrac{\partial v_R}{\partial z} \\ \dfrac{\partial v_\vartheta}{\partial R} & \dfrac{1}{R}\left(\dfrac{\partial v_\vartheta}{\partial \vartheta} + v_R\right) & \dfrac{\partial v_\vartheta}{\partial z} \\ \dfrac{\partial v_z}{\partial R} & \dfrac{1}{R}\dfrac{\partial v_z}{\partial \vartheta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix}. \tag{3.10b}$$

According to equations (1.88) the velocity gradient can be resolved into symmetric and skew parts:

$$\boldsymbol{l} = \boldsymbol{d} + \boldsymbol{\Omega},$$
$$\boldsymbol{d} = \frac{1}{2}\left(\boldsymbol{l} + \boldsymbol{l}^T\right) = \frac{1}{2}\left(\mathbf{v}\circ\nabla + \nabla\circ\mathbf{v}\right), \quad \boldsymbol{\Omega} = \frac{1}{2}\left(\boldsymbol{l} - \boldsymbol{l}^T\right) = \frac{1}{2}\left(\mathbf{v}\circ\nabla - \nabla\circ\mathbf{v}\right), \tag{3.11a}$$

$$\begin{aligned} \ell_{pq} &= d_{pq} + \Omega_{pq}, \\ d_{pq} &= \frac{1}{2}\left(\ell_{pq} + \ell_{qp}\right) = \ell_{(pq)} = \frac{1}{2}\left(v_{p,q} + v_{q,p}\right), \\ \Omega_{pq} &= \frac{1}{2}\left(\ell_{pq} - \ell_{qp}\right) = \ell_{[pq]} = \frac{1}{2}\left(v_{p,q} - v_{q,p}\right). \end{aligned} \tag{3.11b}$$

Here $\boldsymbol{d}$ is the strain rate tensor and $\boldsymbol{\Omega}$ is the spin tensor. The latter is sometimes called the rate of rotation tensor or vorticity tensor [90] – we prefer, however, the expression spin tensor.

Let us denote the axial vector of the skew spin tensor by $\boldsymbol{\omega}$. The linearized velocity distribution (3.6a) at the spatial point P can be rewritten if we take into account the additive resolution (3.11) and apply relationship (1.90):

$$\mathrm{d}\mathbf{v} = \boldsymbol{l}\cdot\mathrm{d}\mathbf{x} = \boldsymbol{d}\cdot\mathrm{d}\mathbf{x} + \boldsymbol{\Omega}\cdot\mathrm{d}\mathbf{x} = \boldsymbol{d}\cdot\mathrm{d}\mathbf{x} + \boldsymbol{\omega}\times\mathrm{d}\mathbf{x}; \quad \boldsymbol{\omega} = -\frac{1}{2}\mathbf{v}\times\nabla. \tag{3.12}$$

The vector $\boldsymbol{\omega}$ is called angular velocity vector. The vector $\mathbf{w} = 2\boldsymbol{\omega} = \nabla\times\mathbf{v}$ is the spin or vorticity vector.

Remark 3.2: The expression $\boldsymbol{\Omega}\cdot\mathrm{d}\mathbf{x} = \boldsymbol{\omega}\times\mathrm{d}\mathbf{x}$ in equation (3.12) describe the velocity change caused by the angular velocity vector in the elementary neighborhood of the material point P. On the other hand $\boldsymbol{d}\cdot\mathrm{d}\mathbf{x}$ is the velocity change due to the rate of deformation tensor and it reflects that the deformations depend on time. For rigid bodies this term is always zero.

According to their definitions the tensor fields $\boldsymbol{l}$, $\boldsymbol{d}$, $\boldsymbol{\Omega}$ and the vector field $\boldsymbol{\omega}$ are all spatial tensors which depend on location and time.

Exercise 3.3: Determine the matrices of the strain rate tensor and the spin tensor in the coordinate system (xyz).

It is not too difficult to verify using equations (3.7), (3.11) and (1.94) that

$$\underline{\boldsymbol{d}} = \left[\ \underline{\boldsymbol{d}}_x \ \middle|\ \underline{\boldsymbol{d}}_y \ \middle|\ \underline{\boldsymbol{d}}_z\ \right] = \begin{bmatrix} d_{xx} & d_{xy} & d_{xz} \\ d_{yx} & d_{yy} & d_{yz} \\ d_{zx} & d_{zy} & d_{zz} \end{bmatrix} =$$
$$= \begin{bmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial v_x}{\partial y} + \dfrac{\partial v_y}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_x}{\partial z} + \dfrac{\partial v_z}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_y}{\partial x} + \dfrac{\partial v_x}{\partial y}\right) & \dfrac{\partial v_y}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v_y}{\partial z} + \dfrac{\partial v_z}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_z}{\partial x} + \dfrac{\partial v_x}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_z}{\partial y} + \dfrac{\partial v_y}{\partial z}\right) & \dfrac{\partial v_z}{\partial z} \end{bmatrix} \tag{3.13}$$

and

$$\underline{\boldsymbol{\Omega}} = \begin{bmatrix} \Omega_{xx} & \Omega_{xy} & \Omega_{xz} \\ \Omega_{yx} & \Omega_{yy} & \Omega_{yz} \\ \Omega_{zx} & \Omega_{zy} & \Omega_{zz} \end{bmatrix} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) & 0 \end{bmatrix}. \tag{3.14}$$

From equation (3.14) we can also see, in accordance with (1.94), how the components of $\boldsymbol{\omega}$ are related to the velocity components.

Exercise 3.4: Find the matrix of the strain rate tensor in the cylindrical coordinate system $(R\vartheta z)$.

A comparison of equations (3.10b) and (3.11) yields

$$\underset{(R\vartheta z)}{\underline{\boldsymbol{d}}} = \left[\underset{(R\vartheta z)}{\boldsymbol{d}_R} \;\middle|\; \underset{(R\vartheta z)}{\boldsymbol{d}_\vartheta} \;\middle|\; \underset{(R\vartheta z)}{\boldsymbol{d}_z} \right] = \begin{bmatrix} d_{RR} & d_{R\vartheta} & d_{Rz} \\ d_{\vartheta R} & d_{\vartheta\vartheta} & d_{\vartheta z} \\ d_{zR} & d_{z\vartheta} & d_{zz} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial v_R}{\partial R} & \frac{1}{2}\left(\frac{1}{R}\frac{\partial v_R}{\partial \vartheta} - \frac{v_\vartheta}{R} + \frac{\partial v_\vartheta}{\partial R}\right) & \frac{1}{2}\left(\frac{\partial v_R}{\partial z} + \frac{\partial v_z}{\partial R}\right) \\ \frac{1}{2}\left(\frac{\partial v_\vartheta}{\partial R} + \frac{1}{R}\frac{\partial v_R}{\partial \vartheta} - \frac{v_\vartheta}{R}\right) & \frac{1}{R}\left(\frac{\partial v_\vartheta}{\partial \vartheta} + v_R\right) & \frac{1}{2}\left(\frac{\partial v_\vartheta}{\partial z} + \frac{1}{R}\frac{\partial v_z}{\partial \vartheta}\right) \\ \frac{1}{2}\left(\frac{\partial v_z}{\partial R} + \frac{\partial v_R}{\partial z}\right) & \frac{1}{2}\left(\frac{1}{R}\frac{\partial v_z}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial z}\right) & \frac{\partial v_z}{\partial z} \end{bmatrix}. \tag{3.15}$$

Exercise 3.5: What is the formula for the angular velocity vector in the cylindrical coordinate system $(R\vartheta z)$?

Making use of equations (3.10a) for $\boldsymbol{l}$ and (1.91) for the axial vector $\mathbf{s}^a$ we find that

$$\boldsymbol{\omega} = -\frac{1}{2}\mathbf{v} \times \nabla = -\frac{1}{2}\left(\frac{\partial \mathbf{v}}{\partial R} \times \mathbf{e}_R + \frac{1}{R}\frac{\partial \mathbf{v}}{\partial \vartheta} \times \mathbf{e}_\vartheta + \frac{\partial \mathbf{v}}{\partial z} \times \mathbf{e}_z\right). \tag{3.16}$$

3.3. Time rates of the strain measures

3.3.1. Introductory remarks. Sections 2.2.3 and 2.5.1 in Chapter 2 are concerned among others with the various strain measures. Due to the motion of the continuum element the length and direction of the line element vectors (the line element), the angles they form, the vectorial and scalar surface elements and the volume element change all with time. There arises the question how to determine the time rates of these changes. The present section is devoted to the determination of these velocities.

3.3.2. Time rates of the deformation gradients. On the basis of (3.1) we may write

$$\left.\frac{\mathrm{d}(\mathrm{d}\mathbf{x})}{\mathrm{d}t}\right|_{(X)} = \frac{\partial(\mathrm{d}\mathbf{x})}{\partial t} = \mathrm{d}\mathbf{v}\,. \tag{3.17}$$

This quantity – as the applied notation shows – is the velocity difference between the endpoints of the line element vector $\mathrm{d}\mathbf{x}$ with linear approximation. Upon substitution of (2.14), (2.30) and (3.6b) into (3.17) we arrive at the earlier result (3.6a):

$$\mathrm{d}\mathbf{v} = \frac{\partial}{\partial t}\mathrm{d}\mathbf{x} = \frac{\partial}{\partial t}\boldsymbol{F}\cdot\mathrm{d}\mathbf{X} = \underset{\boldsymbol{F}=\mathbf{x}\circ\nabla^{\circ}}{\uparrow} = \Big[\Big(\frac{\partial}{\partial t}\mathbf{x}\Big)\circ\underbrace{\nabla^{\circ}}_{\nabla\cdot\boldsymbol{F}}\Big]\cdot\mathrm{d}\mathbf{X} =$$

$$= \left[\Big(\frac{\partial}{\partial t}\mathbf{x}\Big)\circ\nabla\right]\cdot\underbrace{\boldsymbol{F}\cdot\mathrm{d}\mathbf{X}}_{\mathrm{d}\mathbf{x}} = (\mathbf{v}\circ\nabla)\cdot\mathrm{d}\mathbf{x} = \boldsymbol{l}\cdot\mathrm{d}\mathbf{x}\,. \tag{3.18}$$

We can now rewrite the velocity difference (3.18) into the form

$$\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}\mathbf{x}) = (\mathrm{d}\mathbf{x})^{\bullet} = \mathrm{d}\mathbf{v} = \boldsymbol{l}\cdot\mathrm{d}\mathbf{x} \qquad (\mathrm{d}x_i)^{\bullet} = \mathrm{d}v_i = l_{ip}\,\mathrm{d}x_p \tag{3.19}$$

if on the basis of (3.17) we introduce the following notational convention for the time rates of the various strain measures:

$$\left.\frac{\mathrm{d}(\ldots)}{\mathrm{d}t}\right|_{(X)} = \frac{\mathrm{D}}{\mathrm{D}t}(\cdots) = (\ldots)^{\bullet}\,. \tag{3.20}$$

With $\boldsymbol{l}$ we may determine the time rate of the deformation gradient. On the basis of (2.14) and (3.19) we have

$$\mathrm{d}\mathbf{v} = \boldsymbol{l}\cdot\mathrm{d}\mathbf{x} = \boldsymbol{l}\cdot\boldsymbol{F}\cdot\mathrm{d}\mathbf{X}\,, \quad \text{and} \quad \mathrm{d}\mathbf{v} = (\boldsymbol{F}\cdot\mathrm{d}\mathbf{X})^{\bullet} = (\boldsymbol{F})^{\bullet}\cdot\mathrm{d}\mathbf{X}$$

from where it follows that

$$\boxed{\begin{aligned} &(\boldsymbol{F})^{\bullet} = \boldsymbol{l}\cdot\boldsymbol{F}\,, && \left(\boldsymbol{F}^{T}\right)^{\bullet} = \boldsymbol{F}^{T}\cdot\boldsymbol{l}^{T}\,;\\ &(F_{\ell A})^{\bullet} = l_{\ell k}F_{kA}\,, && (F_{A\ell})^{\bullet} = F_{Ak}l_{k\ell}\,. \end{aligned}} \tag{3.21a}$$

Since the unit tensor (identity tensor) does not change with time (it is constant) the time rate of the product $\boldsymbol{F}\cdot\boldsymbol{F}^{-1} = \boldsymbol{1}$ is the zero tensor. Hence

$$(\boldsymbol{F})^{\bullet}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}\cdot\left(\boldsymbol{F}^{-1}\right)^{\bullet} = \boldsymbol{l}\cdot\underbrace{\boldsymbol{F}\cdot\boldsymbol{F}^{-1}}_{\boldsymbol{1}} + \boldsymbol{F}\cdot\left(\boldsymbol{F}^{-1}\right)^{\bullet} = \boldsymbol{0}\,.$$

Dot multiply this equation from left by $\boldsymbol{F}^{-1}$. After rearranging this result we get the time rate of the inverse deformation gradient:

$$\boxed{\begin{aligned} &\left(\boldsymbol{F}^{-1}\right)^{\bullet} = -\boldsymbol{F}^{-1}\cdot\boldsymbol{l}\,, && \left(\boldsymbol{F}^{-T}\right)^{\bullet} = -\boldsymbol{l}^{T}\cdot\boldsymbol{F}^{-T}\,;\\ &\left(F^{-1}_{Bk}\right)^{\bullet} = -F^{-1}_{Br}\cdot l_{rk}\,, && \left(F^{-1}_{kB}\right)^{\bullet} = -l_{kr}F^{-1}_{rB}\,. \end{aligned}} \tag{3.21b}$$

When determining the time rates of the various strain measures we shall need the following time derivatives (3.19), (3.21a) and (3.21b).

3.3.3. Time rates of the line elements and the angles they form. Consider now the dot product of the two line element vectors shown in Figure 2.4:

$$\mathrm{d}\mathbf{x}_{\mathrm{I}} \cdot \mathrm{d}\mathbf{x}_{\mathrm{II}} = \mathrm{d}s_{\mathrm{I}}\,\mathrm{d}s_{\mathrm{II}}\,\cos\alpha\,.$$

If we utilize (3.19), (2.41b) and (3.11) we obtain

$$\begin{aligned}(\mathrm{d}\,\mathbf{x}_{\mathrm{I}} \cdot \mathrm{d}\,\mathbf{x}_{\mathrm{II}})^{\bullet} &= (\mathrm{d}\,\mathbf{x}_{\mathrm{I}})^{\bullet} \cdot \mathrm{d}\,\mathbf{x}_{\mathrm{II}} + \mathrm{d}\,\mathbf{x}_{\mathrm{I}} \cdot (\mathrm{d}\,\mathbf{x}_{\mathrm{II}})^{\bullet} = \mathrm{d}\mathbf{v}_{\mathrm{I}} \cdot \mathrm{d}\,\mathbf{x}_{\mathrm{II}} + \mathrm{d}\mathbf{x}_{\mathrm{I}} \cdot \mathrm{d}\,\mathbf{v}_{\mathrm{II}} = \\ &= \mathrm{d}\mathbf{x}_{\mathrm{I}} \cdot \boldsymbol{l}^T \cdot \mathrm{d}\mathbf{x}_{\mathrm{II}} + \mathrm{d}\mathbf{x}_{\mathrm{I}} \cdot \boldsymbol{l} \cdot \mathrm{d}\mathbf{x}_{\mathrm{II}} = 2\,\mathbf{e}_{\mathrm{I}} \cdot \frac{1}{2}\left(\boldsymbol{l} + \boldsymbol{l}^T\right) \cdot \mathbf{e}_{\mathrm{II}}\,\mathrm{d}s_{\mathrm{I}}\,\mathrm{d}s_{\mathrm{II}} = \\ &= 2\,\mathbf{e}_{\mathrm{I}} \cdot \boldsymbol{d} \cdot \mathbf{e}_{\mathrm{II}}\,\mathrm{d}s_{\mathrm{I}}\,\mathrm{d}s_{\mathrm{II}}\end{aligned} \tag{3.22a}$$

for the time derivative on the left side. As regards the right side we may write

$$(\mathrm{d}s_{\mathrm{I}}\,\mathrm{d}s_{\mathrm{II}}\,\cos\alpha)^{\bullet} = \left[(\mathrm{d}s_{\mathrm{I}})^{\bullet}\,\mathrm{d}s_{\mathrm{II}} + \mathrm{d}s_{\mathrm{I}}\,(\mathrm{d}s_{\mathrm{II}})^{\bullet}\right]\cos\alpha - \mathrm{d}s_{\mathrm{I}}\,\mathrm{d}s_{\mathrm{II}}\,(\alpha)^{\bullet}\sin\alpha\,. \tag{3.22b}$$

For $\alpha = 0$ a comparison of (3.22a) and (3.22b) yields

$$\boxed{\frac{(\mathrm{d}s)^{\bullet}}{\mathrm{d}s} = \mathbf{e} \cdot \boldsymbol{d} \cdot \mathbf{e} \qquad (\mathbf{e}_{\mathrm{I}} = \mathbf{e}_{\mathrm{II}} = \mathbf{e})} \tag{3.23}$$

which is the time rate of the line element Equation (3.23) can be rewritten if we take definition (2.35) of the stretch ratio λ^e into account:

$$\frac{(\mathrm{d}s)^{\bullet}}{\mathrm{d}s} = \frac{(\mathrm{d}s)^{\bullet}}{\mathrm{d}s^{\circ}}\frac{\mathrm{d}s^{\circ}}{\mathrm{d}s} = \frac{(\lambda^e)^{\bullet}}{\lambda^e} = (\ln\lambda^e)^{\bullet}\,. \tag{3.24}$$

Hence

$$\boxed{\frac{(\mathrm{d}s)^{\bullet}}{\mathrm{d}s} = \frac{\mathrm{D}}{\mathrm{D}t}(\ln\lambda^e) = (\ln\lambda^e)^{\bullet} = \mathbf{e} \cdot \boldsymbol{d} \cdot \mathbf{e} = e_k\,d_{k\ell}\,e_\ell\,.} \tag{3.25}$$

Remembering that (3.22a) and (3.22b) are equal for $\alpha \neq 0$ we have

$$\boxed{\begin{aligned}&\left[\frac{(\mathrm{d}s_{\mathrm{I}})^{\bullet}}{\mathrm{d}s_{\mathrm{I}}} + \frac{(\mathrm{d}s_{\mathrm{II}})^{\bullet}}{\mathrm{d}s_{\mathrm{II}}}\right]\cos\alpha - (\alpha)^{\bullet}\sin\alpha = \\ &\qquad = \left[\frac{(\lambda^e_{\mathrm{I}})^{\bullet}}{\lambda^e_{\mathrm{I}}} + \frac{(\lambda^e_{\mathrm{II}})^{\bullet}}{\lambda^e_{\mathrm{II}}}\right]\cos\alpha - (\alpha)^{\bullet}\sin\alpha = 2\,\mathbf{e}_{\mathrm{I}} \cdot \boldsymbol{d} \cdot \mathbf{e}_{\mathrm{II}}\,.\end{aligned}} \tag{3.26}$$

This equation can easily be solved for $(\alpha)^{\bullet}$ which is the time rate of the angle α constituted by the line element vectors.

Assume now that $\mathbf{e}_{\mathrm{I}} \cdot \mathbf{e}_{\mathrm{II}} = 0$. Then $\alpha = \alpha^{\circ} - \gamma_{12} = 90^{\circ}$, where γ_{12} is the angle change which is positive if α° becomes smaller. Since α° is independent of time (3.26) gets simpler:

$$(\gamma_{12})^{\bullet} = -\,(\alpha)^{\bullet} = 2\,\mathbf{e}_{\mathrm{I}} \cdot \boldsymbol{d} \cdot \mathbf{e}_{\mathrm{II}}\,. \tag{3.27}$$

Exercise 3.6: Find the time rates of the axial strains $\varepsilon^{\circ e}$ and ε^{e}.

With (2.36) and (3.25) we may write

$$(\varepsilon^{\circ e})^{\bullet} = (\lambda^e - 1)^{\bullet} = (\lambda^e)^{\bullet} = \lambda^e\,(\ln\lambda^e)^{\bullet} = \lambda^e\mathbf{e} \cdot \boldsymbol{d} \cdot \mathbf{e} = (1 + \varepsilon^{\circ e})\,\mathbf{e} \cdot \boldsymbol{d} \cdot \mathbf{e} \tag{3.28a}$$

which is the first of the derivatives we look for. As regards the second using (2.37) we get

$$(\varepsilon^e)^{\boldsymbol{\cdot}} = \left(\frac{\varepsilon^{oe}}{\lambda^e}\right)^{\boldsymbol{\cdot}} = \frac{(\varepsilon^{oe})^{\boldsymbol{\cdot}}}{\lambda^e} - \frac{\varepsilon^{oe}}{(\lambda^e)^2}(\lambda^e)^{\boldsymbol{\cdot}} = \mathbf{e}\cdot\boldsymbol{d}\cdot\mathbf{e} - \varepsilon^e\,\mathbf{e}\cdot\boldsymbol{d}\cdot\mathbf{e} =$$
$$= (1-\varepsilon^e)\,\mathbf{e}\cdot\boldsymbol{d}\cdot\mathbf{e} = \frac{1}{\lambda^{oe}}\,\mathbf{e}\cdot\boldsymbol{d}\cdot\mathbf{e} = \frac{1}{1+\varepsilon^{oe}}\,\mathbf{e}\cdot\boldsymbol{d}\cdot\mathbf{e}\,. \qquad (3.28\text{b})$$

Exercise 3.7: Determine the strain rate tensor both in the Cartesian coordinate system (xyz) and in the cylindrical coordinate system $(R\vartheta z)$. Use equations (3.25) and (3.27) for points of departure.

On the basis of (3.25) and (3.27)

$$d_{mm} = \mathbf{e}_m\cdot\boldsymbol{d}\cdot\mathbf{e}_m = (\ln\lambda_m)^{\boldsymbol{\cdot}} \qquad m = x, y, z$$
$$d_{mn} = \mathbf{e}_m\cdot\boldsymbol{d}\cdot\mathbf{e}_n = (\gamma_{mn})^{\boldsymbol{\cdot}} \qquad m, n = x, y, z;\quad m\neq n$$

are the diagonal and off-diagonal elements of $\boldsymbol{d}$. Consequently,

$$\underline{\mathbf{d}} = \left[\,\underline{\mathbf{d}}_x\,\middle|\,\underline{\mathbf{d}}_y\,\middle|\,\underline{\mathbf{d}}_z\,\right] = \begin{bmatrix} d_{xx} & d_{xy} & d_{xz}\\ d_{yx} & d_{yy} & d_{yz}\\ d_{zx} & d_{zy} & d_{zz}\end{bmatrix} = \begin{bmatrix} (\ln\lambda_x)^{\boldsymbol{\cdot}} & (\gamma_{xy})^{\boldsymbol{\cdot}} & (\gamma_{xz})^{\boldsymbol{\cdot}}\\ (\gamma_{yx})^{\boldsymbol{\cdot}} & (\ln\lambda_y)^{\boldsymbol{\cdot}} & (\gamma_{yz})^{\boldsymbol{\cdot}}\\ (\gamma_{zx})^{\boldsymbol{\cdot}} & (\gamma_{zy})^{\boldsymbol{\cdot}} & (\ln\lambda_z)^{\boldsymbol{\cdot}}\end{bmatrix} \qquad (3.29)$$

is the matrix of $\boldsymbol{d}$ in the Cartesian coordinate system. In the cylindrical coordinate system we obtain in the same way that

$$\underset{(R\vartheta z)}{\underline{\mathbf{d}}} = \left[\,\underline{\mathbf{d}}_R\,\middle|\,\underline{\mathbf{d}}_\vartheta\,\middle|\,\underline{\mathbf{d}}_z\,\right] = \begin{bmatrix} d_{RR} & d_{R\vartheta} & d_{Rz}\\ d_{\vartheta R} & d_{\vartheta\vartheta} & d_{\vartheta z}\\ d_{zR} & d_{z\vartheta} & d_{zz}\end{bmatrix} = \begin{bmatrix} (\ln\lambda_R)^{\boldsymbol{\cdot}} & (\gamma_{R\vartheta})^{\boldsymbol{\cdot}} & (\gamma_{Rz})^{\boldsymbol{\cdot}}\\ (\gamma_{\vartheta R})^{\boldsymbol{\cdot}} & (\ln\lambda_\vartheta)^{\boldsymbol{\cdot}} & (\gamma_{\vartheta z})^{\boldsymbol{\cdot}}\\ (\gamma_{zR})^{\boldsymbol{\cdot}} & (\gamma_{z\vartheta})^{\boldsymbol{\cdot}} & (\ln\lambda_z)^{\boldsymbol{\cdot}}\end{bmatrix}. \qquad (3.30)$$

3.3.4. Time rate of change for the volume element. The right side of Figure 2.8 shows the volume element determined by the line element vectors $\mathrm{d}\,\mathbf{x}_{\mathrm{I}}$, $\mathrm{d}\,\mathbf{x}_{\mathrm{II}}$, $\mathrm{d}\,\mathbf{x}_{\mathrm{III}}$ in the current configuration:

$$\mathrm{d}V = [\mathrm{d}\,\mathbf{x}_{\mathrm{I}}\;\mathrm{d}\,\mathbf{x}_{\mathrm{II}}\;\mathrm{d}\,\mathbf{x}_{\mathrm{III}}] = e_{ijk}\mathrm{d}x_i^I\mathrm{d}x_j^{II}\mathrm{d}x_k^{III}\,. \qquad (3.31)$$

If we apply the product rule of derivation and make use of relation (3.19) we get from here that

$$(\mathrm{d}V)^{\boldsymbol{\cdot}} = e_{ijk}\left[\left(\mathrm{d}x_i^I\right)^{\boldsymbol{\cdot}}\mathrm{d}x_j^{II}\mathrm{d}x_k^{III} + \mathrm{d}x_i^I\left(\mathrm{d}x_j^{II}\right)^{\boldsymbol{\cdot}}\mathrm{d}x_k^{III} + \mathrm{d}x_i^I\mathrm{d}x_j^{II}\left(\mathrm{d}x_k^{III}\right)^{\boldsymbol{\cdot}}\right] =$$
$$= e_{ijk}\left[l_{ip}\mathrm{d}x_p^I\mathrm{d}x_j^{II}\mathrm{d}x_k^{III} + \mathrm{d}x_i^I l_{jq}\mathrm{d}x_q^I\mathrm{d}x_k^{III} + \mathrm{d}x_i^I\mathrm{d}x_j^{II}l_{kr}\mathrm{d}x_r^{III}\right] =$$
$$= e_{ijk}\left[l_{ip}\delta_{jq}\delta_{kr} + \delta_{ip}l_{jq}\delta_{kr} + \delta_{ip}\delta_{jq}l_{kr}\right]\mathrm{d}x_p^I\mathrm{d}x_q^{II}\mathrm{d}x_r^{III}\,. \qquad (3.32\text{a})$$

A comparison of the right side and (1.47) leads to the following result:

$$(\mathrm{d}V)^{\boldsymbol{\cdot}} = e_{ijk}\left[l_{i1}\delta_{j2}\delta_{k3} + \delta_{i1}l_{j2}\delta_{k3} + \delta_{i1}\delta_{j2}l_{k3}\right]\,e_{pqr}\mathrm{d}x_p^I\mathrm{d}x_q^{II}\mathrm{d}x_r^{III} =$$
$$= \left(e_{i23}l_{i1} + e_{1j3}l_{j2} + e_{12k}l_{k3}\right)\,\mathrm{d}V = \left(l_{11}+l_{22}+l_{33}\right)\,\mathrm{d}V = l_I\mathrm{d}V = d_I\mathrm{d}V. \qquad (3.32\text{b})$$

Hence

$$\boxed{(\mathrm{d}V)^{\boldsymbol{\cdot}} = d_I\,\mathrm{d}V} \qquad (3.33)$$

is the time rate of change for the volume element.

In direct notation the time derivative of

$$\mathrm{d}V = [\mathrm{d}\,\mathbf{x}_\mathrm{I}\;\mathrm{d}\,\mathbf{x}_\mathrm{II}\;\mathrm{d}\,\mathbf{x}_\mathrm{III}] = \underbrace{[\mathbf{e}_\mathrm{I}\;\mathbf{e}_\mathrm{II}\;\mathbf{e}_\mathrm{III}]}_{e^o}\,\mathrm{d}s_\mathrm{I}\;\mathrm{d}s_\mathrm{II}\;\mathrm{d}s_\mathrm{III} \tag{3.34}$$

results in the same formula

$$\begin{aligned}
&(\mathrm{d}V)^{\boldsymbol{\cdot}} = \\
&= (\mathrm{d}\,\mathbf{x}_\mathrm{II}\times\mathrm{d}\,\mathbf{x}_\mathrm{III})\cdot(\mathrm{d}\,\mathbf{x}_\mathrm{I})^{\boldsymbol{\cdot}} + (\mathrm{d}\,\mathbf{x}_\mathrm{III}\times\mathrm{d}\,\mathbf{x}_\mathrm{I})\cdot(\mathrm{d}\,\mathbf{x}_\mathrm{II})^{\boldsymbol{\cdot}} + (\mathrm{d}\,\mathbf{x}_\mathrm{I}\times\mathrm{d}\,\mathbf{x}_\mathrm{II})\cdot(\mathrm{d}\,\mathbf{x}_\mathrm{III})^{\boldsymbol{\cdot}} = \\
&= \mathrm{d}s_\mathrm{I}\;\mathrm{d}s_\mathrm{II}\;\mathrm{d}s_\mathrm{III}\,[(\mathbf{e}_\mathrm{II}\times\mathbf{e}_\mathrm{III})\cdot\boldsymbol{l}\cdot\mathbf{e}_\mathrm{I} + (\mathbf{e}_\mathrm{III}\times\mathbf{e}_\mathrm{I})\cdot\boldsymbol{l}\cdot\mathbf{e}_\mathrm{II} + (\mathbf{e}_\mathrm{I}\times\mathbf{e}_\mathrm{II})\cdot\boldsymbol{l}\cdot\mathbf{e}_\mathrm{III}] = \\
&= \underbrace{e_o\,(\mathrm{d}s_\mathrm{I}\;\mathrm{d}s_\mathrm{II}\;\mathrm{d}s_\mathrm{III})}_{\mathrm{d}V}\;\underbrace{[\mathbf{e}^*_\mathrm{I}\cdot\boldsymbol{l}\cdot\mathbf{e}_\mathrm{I} + \mathbf{e}^*_\mathrm{II}\cdot\boldsymbol{l}\cdot\mathbf{e}_\mathrm{II} + \mathbf{e}^*_\mathrm{III}\cdot\boldsymbol{l}\cdot\mathbf{e}_\mathrm{III}]}_{l_I = d_I} = d_I\,\mathrm{d}V\,,
\end{aligned}$$

where we have taken into account (c) the cyclic interchangeability of the mixed products, (b) formulae (1.21) for the dual base vectors and (d) relation (1.125a) for the first scalar invariant ($\mathbf{e}_\mathrm{I}$, $\mathbf{e}_\mathrm{II}$, $\mathbf{e}_\mathrm{III}$ corresponds to $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$).

We may rewrite equation (3.33) if we make use of relation (2.94) which defines the volume ratio. We get

$$\frac{(\mathrm{d}V)^{\boldsymbol{\cdot}}}{\mathrm{d}V} = \frac{(\mathrm{d}V)^{\boldsymbol{\cdot}}}{\mathrm{d}V^\circ}\frac{\mathrm{d}V^\circ}{\mathrm{d}V} = \frac{(\lambda_V)^{\boldsymbol{\cdot}}}{\lambda_V} = (\ln\lambda_V)^{\boldsymbol{\cdot}}\;.$$

or

$$\boxed{\frac{(\mathrm{d}V)^{\boldsymbol{\cdot}}}{\mathrm{d}V} = (\ln\lambda_V)^{\boldsymbol{\cdot}} = d_I = \mathbf{v}\cdot\nabla} \tag{3.35}$$

which is the rate of change of a volume element divided by the volume element itself.

Since $\mathrm{d}V = J\mathrm{d}V^\circ$ it also holds that

$$(\mathrm{d}V)^{\boldsymbol{\cdot}} = \underline{(J)^{\boldsymbol{\cdot}}}\,\mathrm{d}V^\circ = d_I\mathrm{d}V = \underline{d_I J}\,\mathrm{d}V^\circ,$$

where the underlined parts are equal to each other. Hence,

$$\boxed{(J)^{\boldsymbol{\cdot}} = d_I\,J = J\,(\mathbf{v}\cdot\nabla)\;.} \tag{3.36}$$

This expression is known as Euler's formula.

3.3.5. Time derivatives of the vectorial and scalar surface elements. The vectorial and scalar surface elements in the current configuration are shown in the right side of Figure 2.7. On the basis of the figure

$$\mathrm{d}\mathbf{A} = \mathrm{d}\,\mathbf{x}_\mathrm{I}\times\mathrm{d}\,\mathbf{x}_\mathrm{II}\,,\qquad \mathrm{d}A_i = e_{ijk}\,\mathrm{d}x^\mathrm{I}_j\,\mathrm{d}x^\mathrm{II}_k \tag{3.37}$$

is the vectorial surface element. Its time derivative is given by the following equation:

$$(\mathrm{d}A_r)^{\boldsymbol{\cdot}} = e_{ijk}\left[\left(\mathrm{d}x^\mathrm{I}_j\right)^{\boldsymbol{\cdot}}\,\mathrm{d}x^\mathrm{II}_k + \mathrm{d}x^\mathrm{I}_j\,\left(\mathrm{d}x^\mathrm{II}_k\right)^{\boldsymbol{\cdot}}\right].$$

We can manipulate the right side of this equation into a more suitable form if we utilize (3.19) by taking into account that the Kronecker delta is an index renaming operator and if we enlarge the result obtained by the underlined terms the sum of which is equal to zero:

$$(\mathrm{d}A_i)^{\bullet} = e_{ijk}\left[\underline{l_{ip}\delta_{jq}\delta_{kr}} + \delta_{ip}l_{jq}\delta_{kr} + \delta_{ip}\delta_{jq}l_{kr} - \underline{l_{ip}\delta_{jq}\delta_{kr}}\right]\mathrm{d}x_q^{\mathrm{I}}\,\mathrm{d}x_r^{\mathrm{II}}\,.$$

If we recall (3.32) we can make the first three terms on the right side simpler:

$$e_{ijk}\left[l_{ip}\delta_{jq}\delta_{kr} + \delta_{ip}l_{jq}\delta_{kr} + \delta_{ip}\delta_{jq}l_{kr}\right]\mathrm{d}x_q^{\mathrm{I}}\,\mathrm{d}x_r^{\mathrm{II}} = d_{\mathrm{I}}\,e_{pqr}\mathrm{d}x_q^{\mathrm{I}}\,\mathrm{d}x_r^{\mathrm{II}} = d_{\mathrm{I}}\,\mathrm{d}A_p\,.$$

By applying resolution (3.11) of the velocity gradient to the last term we may write

$$\begin{aligned}-\,e_{ijk}l_{ip}\delta_{jq}\delta_{kr}\mathrm{d}x_q^{\mathrm{I}}\,\mathrm{d}x_r^{\mathrm{II}} &= -l_{ip}e_{iqr}\mathrm{d}x_q^{\mathrm{I}}\,\mathrm{d}x_r^{\mathrm{II}} = -l_{ip}\mathrm{d}A_i = \\ &= -\left(d_{ip} + \Omega_{ip}\right)\mathrm{d}A_i = -d_{pi}\mathrm{d}A_i + \Omega_{pi}\mathrm{d}A_i\,.\end{aligned}$$

After gathering the partial results we get the time derivative of the surface element vector:

$$\boxed{(\mathrm{d}A_i)^{\bullet} = \left(d_{\mathrm{I}}\delta_{ip} - d_{ip}\right)\mathrm{d}A_p + \Omega_{ip}\mathrm{d}A_p} \tag{3.38}$$

since $\Omega_{ip} = -\Omega_{pi}$ and $d_{pi} = d_{ip}$.

In direct notation the Nanson formula (2.89b) is our point of departure. Its time derivative is given by

$$(\mathrm{d}\mathbf{A})^{\bullet} = (J)^{\bullet}\,\boldsymbol{F}^{-T}\cdot\mathrm{d}\,\mathbf{A}^{\circ} + J\left(\boldsymbol{F}^{-T}\right)^{\bullet}\cdot\mathrm{d}\,\mathbf{A}^{\circ}\,.$$

Substituting the derivatives (3.36) and (3.21b) this expression becomes

$$(\mathrm{d}\mathbf{A})^{\bullet} = d_I\underbrace{J\boldsymbol{F}^{-T}\cdot\mathrm{d}\,\mathbf{A}^{\circ}}_{\mathrm{d}\mathbf{A}} - \boldsymbol{l}^T\cdot\underbrace{J\boldsymbol{F}^{-T}\cdot\mathrm{d}\,\mathbf{A}^{\circ}}_{\mathrm{d}\mathbf{A}} = d_I\,\mathrm{d}\mathbf{A} - \boldsymbol{l}^T\cdot\mathrm{d}\mathbf{A}\,.$$

If we now utilize the additive resolution (3.11) finally we get

$$\boxed{(\mathrm{d}\mathbf{A})^{\bullet} = \left(d_I\,\boldsymbol{1} - \boldsymbol{d}\right)\cdot\mathrm{d}\mathbf{A} + \boldsymbol{\Omega}\cdot\mathrm{d}\mathbf{A} = \left(d_I\,\boldsymbol{1} - \boldsymbol{d}\right)\cdot\mathrm{d}\mathbf{A} + \boldsymbol{\omega}\times\mathrm{d}\mathbf{A}} \tag{3.39}$$

which obviously coincides with (3.38).

Remark 3.3: Expression $\boldsymbol{\Omega}\cdot\mathrm{d}\mathbf{A} = \boldsymbol{\omega}\times\mathrm{d}\mathbf{A}$ in equation (3.39) is the velocity change due to the rotation of the vectorial surface element $\mathrm{d}\mathbf{A}$. On the contrary $(d_I\,\boldsymbol{1} - \boldsymbol{d})\cdot\mathrm{d}\mathbf{A}$ reflects the velocity change caused by the fact that the deformation of the surface element depends on time. When we consider the motion of a rigid body this quantity is always zero.

As regards the time derivative of the scalar surface element $\mathrm{d}A$ identity $(\mathrm{d}A)^2 = \mathrm{d}\mathbf{A}\cdot\mathrm{d}\mathbf{A}$ is our point of departure. If we take the time derivatives of the two sides we get

$$(\mathrm{d}A)^{\bullet}\,\mathrm{d}A = (\mathrm{d}\mathbf{A})^{\bullet}\cdot\mathrm{d}\mathbf{A}\,.$$

Substitute now derivative (3.39) and take into account that the vectorial surface element can be given in the form $\mathrm{d}\mathbf{A} = \mathbf{n}\,\mathrm{d}A$ where $\mathbf{n}$ is the outward unit normal of the surface element. In this way we get the time derivative of the scalar surface element in the following form:

$$\boxed{\frac{(\mathrm{d}A)^{\bullet}}{\mathrm{d}A} = \frac{(\lambda_A)^{\bullet}}{\lambda_A} = d_I - \mathbf{n}\cdot\boldsymbol{d}\cdot\mathbf{n} = \boldsymbol{d}\cdot\cdot(\boldsymbol{1} - \mathbf{n}\circ\mathbf{n})} \tag{3.40}$$

where it has been taken into consideration that

$$\frac{(\mathrm{d}A)^{\bullet}}{\mathrm{d}A} = \frac{(\mathrm{d}A)^{\bullet}}{\mathrm{d}A^{\circ}}\frac{\mathrm{d}A^{\circ}}{\mathrm{d}A} = \frac{(\lambda_A)^{\bullet}}{\lambda_A} = (\ln\lambda_A)^{\bullet} \,.$$

3.3.6. Time derivatives of the strain tensors. With the time derivatives of the deformation gradient (3.21a) and the inverse deformation gradient (3.21b) we can determine the time rate of change of the left Cauchy-Green tensor (2.48):

$$\begin{aligned}\left(\boldsymbol{b}^{-1}\right)^{\bullet} &= \left(\boldsymbol{F}^{-T}\right)^{\cdot}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}^{-T}\cdot\left(\boldsymbol{F}^{-1}\right)^{\bullet} = \\ &= -\boldsymbol{l}^{T}\cdot\underbrace{\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}}_{\boldsymbol{b}^{-1}} - \underbrace{\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}}_{\boldsymbol{b}^{-1}}\cdot\boldsymbol{l} = -\boldsymbol{l}^{T}\cdot\boldsymbol{b}^{-1} - \boldsymbol{b}^{-1}\cdot\boldsymbol{l}\,. \end{aligned} \tag{3.41}$$

If we utilize this result we can also determine the time derivative of the Euler-Almansi strain tensor $\boldsymbol{e}$. To find a more suitable form of the derivative

$$(\boldsymbol{e})^{\bullet} = \left(\frac{1}{2}\left(\boldsymbol{1} - \boldsymbol{b}^{-1}\right)\right)^{\bullet} = -\frac{1}{2}\left(\boldsymbol{b}^{-1}\right)^{\bullet} = \frac{1}{2}\left(\boldsymbol{l}^{T}\cdot\boldsymbol{b}^{-1} + \boldsymbol{b}^{-1}\cdot\boldsymbol{l}\right) \tag{3.42}$$

we got it is worthy of taking the relation $\boldsymbol{b}^{-1} = \boldsymbol{1} - 2\boldsymbol{e}$, which follows from (2.48), into account. We have

$$(\boldsymbol{e})^{\bullet} = \frac{1}{2}\left(\boldsymbol{l} + \boldsymbol{l}^{T}\right) - \boldsymbol{e}\cdot\boldsymbol{l} - \boldsymbol{l}^{T}\cdot\boldsymbol{e} \underset{(3.11)}{\overset{\uparrow}{=}} \boldsymbol{d} - \boldsymbol{e}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{e} - \boldsymbol{e}\cdot\boldsymbol{\Omega} + \boldsymbol{\Omega}\cdot\boldsymbol{e}\,, \tag{3.43a}$$

where $\boldsymbol{\Omega}$ is skew. Hence

$$\boxed{(\boldsymbol{e})^{\bullet} = \boldsymbol{d} - \boldsymbol{e}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{e} - \boldsymbol{e}\times\boldsymbol{\omega} + \boldsymbol{\omega}\times\boldsymbol{e}\,.} \tag{3.43b}$$

is the time derivative of the Euler-Almansi strain tensor.

Remark 3.4: The first part of the derivative (3.43b), i.e., the expression

$$\boxed{\boldsymbol{e}^{\nabla} = \boldsymbol{d} - \boldsymbol{e}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{e} = (\boldsymbol{e})^{\bullet} + (\boldsymbol{e}\times\boldsymbol{\omega} - \boldsymbol{\omega}\times\boldsymbol{e})} \tag{3.44}$$

is called objective Jaumann time derivative of the Euler-Almansi strain tensor [39, p. 1911]. Since (a) $\boldsymbol{e}$ is independent of the rotation tensor $\boldsymbol{R}$ and (b) $\boldsymbol{d}$ is independent of the spin tensor $\boldsymbol{\Omega}$ (or the local angular velocity $\boldsymbol{\omega}$) it follows that the Jaumann time derivative reflects the velocity changes due to pure deformation. On the contrary the effect the local rotation has on the velocity changes appears via the terms $-\boldsymbol{e}\times\boldsymbol{\omega} + \boldsymbol{\omega}\times\boldsymbol{e}$. The material behavior of the continuum should be independent of the local rotations. Hence those equations which relate the stress rates to the deformation rates should contain objective time rates only. Significance of the Jaumann time derivative can be explained by this fact.

REMARK 3.5: Let $\boldsymbol{h}$ be a symmetric spatial tensor which describes a physical state of the continuum in the current configuration. Its Jaumann time derivative can be given on the basis of (3.43) and (3.44) in the following form:

$$\boldsymbol{h}^{\nabla} = (\boldsymbol{h})^{\boldsymbol{\cdot}} + (\boldsymbol{h} \cdot \boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \boldsymbol{h}) = (\boldsymbol{h})^{\boldsymbol{\cdot}} + (\boldsymbol{h} \times \boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{h}) \,. \tag{3.45}$$

REMARK 3.6: Further objective time derivatives were introduced by Oldroyed [42], Cotter-Rivlin [44] and Trusdell [46, 45]. For the tensor $\boldsymbol{h}$ they assume the following forms:

$$\boldsymbol{h}^{\nabla}_{\text{Oldroyod}} = (\boldsymbol{h})^{\boldsymbol{\cdot}} - \boldsymbol{l} \cdot \boldsymbol{h} - \boldsymbol{h} \cdot \boldsymbol{l}^T \,, \tag{3.46}$$

$$\boldsymbol{h}^{\nabla}_{\text{Cotter-Rivlin}} = (\boldsymbol{h})^{\boldsymbol{\cdot}} + \boldsymbol{l}^T \cdot \boldsymbol{h} + \boldsymbol{h} \cdot \boldsymbol{l} \tag{3.47}$$

and

$$\boldsymbol{h}^{\nabla}_{\text{Trusdell}} = (\boldsymbol{h})^{\boldsymbol{\cdot}} - \boldsymbol{l} \cdot \boldsymbol{h} - \boldsymbol{h} \cdot \boldsymbol{l}^T + d_I \boldsymbol{h} \,. \tag{3.48}$$

EXERCISE 3.8: Find the time derivative of the Cauchy strain tensor $\boldsymbol{b}$.

Making use of the time derivative of the deformation gradient (3.21a) from (2.49) we get:

$$(\boldsymbol{b})^{\boldsymbol{\cdot}} = (\boldsymbol{F})^{\boldsymbol{\cdot}} \cdot \boldsymbol{F}^T + \boldsymbol{F} \cdot (\boldsymbol{F}^T)^{\boldsymbol{\cdot}} = \boldsymbol{l} \cdot \underbrace{\boldsymbol{F} \cdot \boldsymbol{F}^T}_{\boldsymbol{b}} + \underbrace{\boldsymbol{F} \cdot \boldsymbol{F}^T}_{\boldsymbol{b}} \cdot \boldsymbol{l}^T = \boldsymbol{l} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{l}^T . \tag{3.49}$$

If we take the time derivative of equation (2.67) we can clarify how the time derivatives of the tensors $\boldsymbol{E}$ and $\boldsymbol{e}$ are related to each other. Upon substitution of relations (3.21a) and (3.43a) into the time derivative

$$\left.\frac{\mathrm{d}\,\boldsymbol{E}}{\mathrm{d}t}\right|_{(X)} = \frac{\partial\,\boldsymbol{E}}{\partial t} = \left(\boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F}\right)^{\boldsymbol{\cdot}} = \left(\boldsymbol{F}^T\right)^{\boldsymbol{\cdot}} \cdot \boldsymbol{e} \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot (\boldsymbol{e})^{\boldsymbol{\cdot}} \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot (\boldsymbol{F})^{\boldsymbol{\cdot}} =$$
$$= \boldsymbol{F}^T \cdot \boldsymbol{l}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \left(\boldsymbol{d} - \boldsymbol{e} \cdot \boldsymbol{l} - \boldsymbol{l}^T \cdot \boldsymbol{e}\right) \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \boldsymbol{l} \cdot \boldsymbol{F}$$

we obtain:

$$\boxed{(\boldsymbol{E})^{\boldsymbol{\cdot}} = \boldsymbol{F}^T \cdot \boldsymbol{d} \cdot \boldsymbol{F} \,.} \tag{3.50a}$$

It follows from here that

$$\boxed{\boldsymbol{d} = \boldsymbol{F}^{-T} \cdot (\boldsymbol{E})^{\boldsymbol{\cdot}} \cdot \boldsymbol{F}^{-1}} \tag{3.50b}$$

which is the pair of equation (3.50a) .

REMARK 3.7: Equation (3.50a) shows that the time derive of the Green-Lagrange strain tensor $(\boldsymbol{E})^{\boldsymbol{\cdot}}$ is a pull-back operation of the strain rate tensor $\boldsymbol{d}$. According to equation (3.50b) the strain rate tensor $\boldsymbol{d}$ is a push-forward operation of the time derivative of the Green-Lagrange strain tensor $(\boldsymbol{E})^{\boldsymbol{\cdot}}$.

3.4. Material time derivatives

The totality of the values a tensor function has in a volume region (regarded for example either in the initial (reference) configuration or in the current configuration) is called a tensor field. It is also worthy of mentioning that a tensor field defined in one of the mentioned volume regions is always considered at the various points of a coordinate system which we call the defining coordinate system. If the tensor changes with time in the defining coordinate system we speak about time dependent tensor field otherwise the tensor field considered is a stationery one. As regards the displacement and deformation states of the continuum Table 1 gathers the various and time dependent tensor fields:

TABLE 1.

Time dependent tensor fields		
Initial configuration	Two point tensors	Current configuration
$\mathbf{u}^\circ\,(X_1, X_2, X_3; t)$		$\mathbf{u}(x_1, x_2, x_3; t),\ \boldsymbol{e}(x_1, x_2, x_3; t)$
$\boldsymbol{C}\,(X_1, X_2, X_3; t)$	$\boldsymbol{F}(x_1, x_2, x_3; t)$	$\boldsymbol{b}(x_1, x_2, x_3; t),\ \boldsymbol{b}^{-1}(x_1, x_2, x_3; t)$
$\mathbf{v}^\circ\,(X_1, X_2, X_3; t)$	$\boldsymbol{F}^{-1}(x_1, x_2, x_3; t)$	$\mathbf{v}(x_1, x_2, x_3; t),\ \boldsymbol{l}(x_1, x_2, x_3; t)$
$\boldsymbol{E}\,(X_1, X_2, X_3; t)$	$\boldsymbol{R}(x_1, x_2, x_3; t)$	$\boldsymbol{d}(x_1, x_2, x_3; t),\ \boldsymbol{\Omega}(x_1, x_2, x_3; t)$
$\boldsymbol{U}\,(X_1, X_2, X_3; t)$		$\boldsymbol{\omega}(x_1, x_2, x_3; t),\ \boldsymbol{v}(x_1, x_2, x_3; t)$

We remark that the two point tensors in Table 1 are regarded in spatial description.

In Sections 3.1 and 3.4 we dealt with the issue of how to determine the time rates of change of some time dependent tensor fields and strain measures. In the sequel we shall supplement and make more precise the results we have got. A time dependent tensor field, for instance the tensor function $\boldsymbol{S}$ – see equation (2.6) – can be given both in material description and in spatial description as well:

$$\boldsymbol{S} = \boldsymbol{\Psi}\,(X_1, X_2, X_3; t)\ , \qquad\qquad \boldsymbol{S} = \boldsymbol{\Phi}\,(x_1, x_2, x_3; t)\ . \tag{3.51}$$

If the tensor function $\boldsymbol{S}$ is given in material description its time derivative is of the form

$$\frac{d}{dt}\boldsymbol{S} = \frac{d}{dt}\boldsymbol{\Psi}\,(X_1, X_2, X_3; t) = \frac{\partial}{\partial t}\boldsymbol{\Psi}\,(X_1, X_2, X_3; t) \tag{3.52a}$$

which reflects the fact that the material coordinates X_1, X_2 and X_3 are independent of time. Time derivative (3.52a) yields the time rate of change with respect to the defining coordinate system – this is now the coordinate system (X_1, X_2, X_3) – at those points of this coordinate system where the tensor is defined.

The meaning of the partial time derivative

$$\frac{\partial}{\partial t}\boldsymbol{S} = \frac{\partial}{\partial t}\boldsymbol{\Phi}\,(x_1, x_2, x_3; t) \tag{3.52b}$$

in spatial description is similar: it gives again the time rate of change with respect to the defining coordinate system – this is now the coordinate system

(x_1, x_2, x_3) – at those points of this coordinate system where the tensor is defined. Time derivative (3.52b) can not be attached to a continuum point because it is considered at a fixed spatial point while the material points are in motion and can, therefore, be found at various spatial points at different points of time. In other words we have not taken into account so far that $x_\ell = \chi_\ell\,(X_1, X_2, X_3; t)$ in (3.52b) by which tensor function $\boldsymbol{\Phi}$ is attached to the material point identified by the coordinate triplet (X_1, X_2, X_3).

$$\begin{aligned} \boldsymbol{S} &= \boldsymbol{\Phi}\,(x_1, x_2, x_3; t) = \\ &= \boldsymbol{\Phi}\left[\chi_1\,(X_1, X_2, X_3; t)\,, \chi_2\,(X_1, X_2, X_3; t)\,, \chi_3\,(X_1, X_2, X_3; t)\,; t\right] . \end{aligned} \tag{3.53}$$

Consequently, the tensor field which represents the time rate of change of the tensor function $\boldsymbol{S}$ within the body in spatial description is of the form:

$$\frac{\mathrm{D}\boldsymbol{S}}{\mathrm{D}t} = \frac{\partial}{\partial t}\boldsymbol{\Phi}\left[\chi_1\,(X_1, X_2, X_3; t)\,, \chi_2\,(X_1, X_2, X_3; t)\,, \chi_3\,(X_1, X_2, X_3; t)\,; t\right] . \tag{3.54}$$

This quantity is referred to as the material time derivative of the tensor function $\boldsymbol{S}$. By taking the relations

$$x_\ell = \chi_\ell\,(X_1, X_2, X_3; t) \qquad \text{and} \qquad v_\ell = \frac{\partial x_\ell}{\partial t} \tag{3.55}$$

into account we can apply the chain rule to determine the material time derivative of $\boldsymbol{S}$:

$$\boxed{\frac{\mathrm{D}\boldsymbol{S}}{\mathrm{D}t} = (\boldsymbol{S})^{\boldsymbol{\cdot}} = \frac{\partial \boldsymbol{\Phi}}{\partial x_\ell}\frac{\partial x_\ell}{\partial t} + \frac{\partial \boldsymbol{\Phi}}{\partial t} = (\boldsymbol{\Phi}\nabla_\ell)\,v_\ell + \frac{\partial \boldsymbol{\Phi}}{\partial t} = (\boldsymbol{\Phi}\circ\nabla)\cdot\mathbf{v} + \frac{\partial \boldsymbol{\Phi}}{\partial t}\,.} \tag{3.56}$$

If $\boldsymbol{S}$ is a scalar field denoted by $\phi\,(x_1, x_2, x_3; t)$ in spatial description equation (3.56) yields:

$$\boxed{\frac{\mathrm{D}\phi}{\mathrm{D}t} = (\phi)^{\boldsymbol{\cdot}} = (\phi\nabla)\cdot\mathbf{v} + \frac{\partial \phi}{\partial t}\,.} \tag{3.57}$$

If $\boldsymbol{S}$ is the velocity field $\mathbf{v}(\mathbf{x}; t) = \mathbf{v}\,(x_1, x_2, x_3; t)$ from equation (3.56) we get the acceleration field also in spatial description:

$$\boxed{\mathbf{a} = \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = (\mathbf{v})^{\boldsymbol{\cdot}} = (\mathbf{v}\circ\nabla)\cdot\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}} \tag{3.58a}$$

or

$$a_k = \frac{\mathrm{D}v_k}{\mathrm{D}t} = (v_k)^{\boldsymbol{\cdot}} = (v_k\nabla_\ell)\,v_\ell + \frac{\partial v_k}{\partial t} = v_{k,\ell}v_\ell + \frac{\partial v_k}{\partial t}\,. \tag{3.58b}$$

Remark 3.8: The velocity vector $\mathbf{v}\,(x_1, x_2, x_3; t)$ is the time rate of change of the time dependent position vector $\mathbf{x} = \boldsymbol{\chi}(X_1, X_2, X_3; t)$ of the material point $\hat{P}(X_1, X_2, X_3)$. Hence $\mathbf{v}\,(x_1, x_2, x_3; t)$ is a material time derivative. The same is valid for the velocity (3.17) (or (3.18)) of the line element vector since the equations referred to here give the velocity changes between the endpoints of the line element vector with a linear approximation. Consequently, each time derivative is a material time derivative if it is obtained by making use of the formulae we have set up for $\mathbf{v}$ and $\mathrm{d}\mathbf{v}$. This means that the following quantities are all material time derivatives:

(a) The time derivatives (3.21a) and (3.21b) of the deformation gradient and the inverse deformation gradient,
(b) the time rate of change of the stretch ratio (3.25),
(c) the time derivative (3.26) of the angle α formed by the line element vectors [or in a special case the derivative (3.27)],
(d) the time rate of change (3.33) of the volume element,
(e) the time derivative of the Jacobian (3.36),
(f) the time derivatives (3.39) and (3.40) of the area element vector $\mathrm{d}\mathbf{A}$ and the scalar area element $\mathrm{d}A$ and finally
(g) the time derivative (3.43) of the Euler-Almansi strain tensor $\boldsymbol{e}$.

In addition to the quantities listed above Section 3.4 contains some further time derivatives. For example $\left(\boldsymbol{b}^{-1}\right)^{\boldsymbol{\cdot}}$ or $(\varepsilon^{\circ e})^{\boldsymbol{\cdot}}$ – see equations (3.42) and (3.28a) for details. However, it follows from their calculation that these quantities are also material time derivatives.

EXERCISE 3.9: Prove the following relation:

$$(\mathbf{v})^{\boldsymbol{\cdot}} \cdot \nabla = (\mathbf{v} \cdot \nabla)^{\boldsymbol{\cdot}} + \boldsymbol{d} \cdot\cdot\, \boldsymbol{d} - \boldsymbol{\omega} \cdot \boldsymbol{\omega}\,. \tag{3.59}$$

In indicial notation

$$(v_k)^{\boldsymbol{\cdot}}{}_{,k} = (v_{k,k})^{\boldsymbol{\cdot}} + d_{k\ell}\, d_{k\ell} - 2\omega_r \omega_r \tag{3.60}$$

is the relation to be proved. Using (3.58b) we can write for the left side of (3.60) that

$$(v_k)^{\boldsymbol{\cdot}}{}_{,k} = \frac{\partial v_{k,k}}{\partial t} + (v_{k,\ell}\, v_\ell)_{,k} = \underbrace{\frac{\partial v_{k,k}}{\partial t} + v_{k,k\ell}\, v_\ell}_{(v_{k,k})^{\boldsymbol{\cdot}}} + v_{k,\ell} v_{\ell,k} = (v_{k,k})^{\boldsymbol{\cdot}} + l_{k\ell} l_{\ell k}\,.$$

The last term on the right side can be manipulated into a more suitable form (a) if we utilize the additive resolution of the velocity gradient (3.11) and then (b) take into account that the inner product of a symmetric and skew tensor is zero. Remembering (1.93) we can also give $\Omega_{k\ell}$ in terms of the angular velocity vector ω_r. The steps are detailed below:

$$\begin{aligned} l_{k\ell} l_{\ell k} &= (d_{k\ell} + \Omega_{k\ell})\,(d_{\ell k} + \Omega_{\ell k}) = d_{k\ell}\, d_{k\ell} - \Omega_{k\ell}\, \Omega_{k\ell} = \\ &= d_{k\ell}\, d_{k\ell} - \underbrace{\epsilon_{k\ell r}\, \epsilon_{k\ell s}}_{2\delta_{rs}}\, \omega_r\, \omega_s = d_{k\ell}\, d_{k\ell} + 2\omega_r\, \omega_r\,. \end{aligned}$$

A comparison of the two partial results shows the validity of relation (3.60).

EXERCISE 3.10: Show that

$$\nabla \times (\mathbf{v})^{\boldsymbol{\cdot}} = 2\,(\boldsymbol{\omega})^{\boldsymbol{\cdot}} + 2\boldsymbol{\omega}\,(\mathbf{v} \cdot \nabla) - (\mathbf{v} \circ \nabla) \cdot 2\boldsymbol{\omega}\,. \tag{3.61}$$

In the formal proof we shall utilize formula (3.58) of the acceleration, (b) relations $2\boldsymbol{\omega} = \nabla \times \mathbf{v}$, $\nabla \cdot \boldsymbol{\omega} = 0$ and (c) expansion rule (1.16) for the triple cross product. Going ahead step by step we may write:

$$\nabla\times (v)^{\boldsymbol{\cdot}} = \frac{\partial \nabla \times \mathbf{v}}{\partial t} + \frac{1}{2} \underbrace{\nabla \times \nabla(\mathbf{v} \cdot \mathbf{v})}_{=\mathbf{0}} + \nabla \times (2\boldsymbol{\omega} \times \mathbf{v}) =$$

$$= \underbrace{\frac{\partial 2\boldsymbol{\omega}}{\partial t} + (2\overset{\downarrow}{\boldsymbol{\omega}} \otimes \nabla) \cdot \mathbf{v}}_{(2\boldsymbol{\omega})^{\boldsymbol{\cdot}}} - (2\overset{\downarrow}{\boldsymbol{\omega}} \otimes \nabla) \cdot \mathbf{v} + 2\boldsymbol{\omega}(\overset{\downarrow}{\mathbf{v}} \cdot \nabla) -$$

$$- (2\boldsymbol{\omega} \cdot \nabla)\overset{\downarrow}{\mathbf{v}} + 2\overset{\downarrow}{\boldsymbol{\omega}}(\mathbf{v} \cdot \nabla) - (2\overset{\downarrow}{\boldsymbol{\omega}} \cdot \nabla)\mathbf{v} =$$

$$= (2\boldsymbol{\omega})^{\boldsymbol{\cdot}} - \underbrace{(2\overset{\downarrow}{\boldsymbol{\omega}} \otimes \nabla) \cdot \mathbf{v} + 2\overset{\downarrow}{\boldsymbol{\omega}}(\mathbf{v} \cdot \nabla)}_{=\mathbf{0}} + 2\boldsymbol{\omega}(\overset{\downarrow}{\mathbf{v}} \cdot \nabla) - (2\overset{\downarrow}{\boldsymbol{\omega}} \cdot \nabla)\mathbf{v} =$$

$$= 2\,(\boldsymbol{\omega})^{\boldsymbol{\cdot}} + 2\boldsymbol{\omega}\,(\mathbf{v} \cdot \nabla) - (\mathbf{v} \otimes \nabla) \cdot 2\boldsymbol{\omega}\ ,$$

where the down arrow shows the quantity the operator ∇ is applied to. The final result of the transformations is the relation we wanted to prove.

3.5. Time derivative of an integral

It is frequently encountered in mechanics of continua that we have to determine the material time derivative of the integral of a spatial tensor field defined in the volume V of the moving continuum. Consider, for example, the arbitrary spatial tensor field $\boldsymbol{\Phi}(x_1, x_2, x_3; t)$ which is defined in the volume region V occupied by the continuum at time t. Its integral is taken in the volume region $V' \subseteq V$ which is either coincides with V or is a part of V:

$$\boldsymbol{J} = \int_{V'} \boldsymbol{\Phi}\, \mathrm{d}V\,. \tag{3.62}$$

If we take relation (3.35) into account and apply the product rule of derivations we get the following formula for the material time derivative of the integral:

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{J}) = (\boldsymbol{J})^{\boldsymbol{\cdot}} = \int_{V'} (\boldsymbol{\Phi})^{\boldsymbol{\cdot}}\ \mathrm{d}V + \int_{V'} \boldsymbol{\Phi}\ \underbrace{(\mathrm{d}V)^{\boldsymbol{\cdot}}}_{(\mathbf{v}\cdot\nabla)\mathrm{d}V} = \int_{V'} \big[(\boldsymbol{\Phi})^{\boldsymbol{\cdot}} + \boldsymbol{\Phi}\,(\mathbf{v} \cdot \nabla)\big]\,\mathrm{d}V\,. \tag{3.63}$$

According to (3.56)

$$(\boldsymbol{\Phi})^{\boldsymbol{\cdot}} + \boldsymbol{\Phi}\,(\mathbf{v} \cdot \nabla) = \frac{\partial \boldsymbol{\Phi}}{\partial t} + (\boldsymbol{\Phi} \circ \nabla) \cdot \mathbf{v} + \boldsymbol{\Phi}\,(\mathbf{v} \cdot \nabla)$$

is the integrand where

$$\big(\overset{\downarrow}{\boldsymbol{\Phi}} \circ \nabla\big) \cdot \mathbf{v} + \boldsymbol{\Phi}\big(\overset{\downarrow}{\mathbf{v}} \cdot \nabla\big) = (\boldsymbol{\Phi} \circ \mathbf{v}) \cdot \nabla\,.$$

Hence,

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{J}) = (\boldsymbol{J})^{\boldsymbol{\cdot}} = \int_{V'} \left[\frac{\partial \boldsymbol{\Phi}}{\partial t} + (\boldsymbol{\Phi} \circ \mathbf{v}) \cdot \nabla\right] \mathrm{d}V} \tag{3.64}$$

is the material time derivative of integral (3.62).

Let us denote the boundary of V' by A' and the outward unit normal on A' by $\mathbf{n}$. It is obvious that the vectorial and scalar surface elements are related to each other via the equation $\mathrm{d}\mathbf{A} = \mathbf{n}\,\mathrm{d}A$. Making use of the divergence theorem

(1.179) we can transform the second volume integral on the right side of (3.64) into a surface integral. The result

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{J}) = (\boldsymbol{J})^{\boldsymbol{\cdot}} = \int_{V'} \frac{\partial \boldsymbol{\Phi}}{\partial t} \mathrm{d}V + \int_{A'} (\boldsymbol{\Phi} \circ \mathbf{v}) \cdot \mathrm{d}\mathbf{A}} \tag{3.65}$$

is the Reynolds[1] transport theorem. [32]. The surface integral

$$\boldsymbol{\Psi} = \int_{V'} (\boldsymbol{\Phi} \circ \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \int_{A'} \boldsymbol{\Phi} \, (\mathbf{v} \cdot \mathbf{n}) \, \mathrm{d}A \tag{3.66}$$

on the right sided is the flux across the surface A' per unit time.

We have a simpler case if the integral of the tensor field $\boldsymbol{\Phi}(x_1, x_2, x_3; t)$ is taken on the mass of the volume region V':

$$\boldsymbol{J} = \int_{V'} \boldsymbol{\Phi} \, \mathrm{d}m = \int_{V'} \boldsymbol{\Phi} \, \rho \, \mathrm{d}V \tag{3.67}$$

in which $\rho(x_1, x_2, x_3; t)$ is the density, $\mathrm{d}m = \rho \, \mathrm{d}V$ is the elementary mass. Then it follows from the principle of mass conservation – see equation (6.6b) – that the material time derivative of the elementary mass vanishes, i.e., $(\mathrm{d}m)^{\boldsymbol{\cdot}} = 0$. Consequently,

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{J}) = (\boldsymbol{J})^{\boldsymbol{\cdot}} = \int_{V'} (\boldsymbol{\Phi})^{\boldsymbol{\cdot}} \, \mathrm{d}m = \int_{V'} (\boldsymbol{\Phi})^{\boldsymbol{\cdot}} \, \rho \, \mathrm{d}V \, .} \tag{3.68}$$

3.6. Problems

PROBLEM 3.1: Given the displacement field of a continuum in spatial description:

$$u_1 = x_1 + \frac{1}{2}(x_1 + x_2)\, e^{-t} - \frac{1}{2}(x_1 - x_2)\, e^{t},$$
$$u_2 = x_2 - \frac{1}{2}(x_1 + x_2)\, e^{-t} - \frac{1}{2}(x_1 - x_2)\, e^{t}\,, \quad u_3 = 0\,.$$

Find the velocity and acceleration fields both in material (Lagrangian) and in spatial (Eulerian) descriptions.

PROBLEM 3.2: Assume that

$$x_1 = X_1 e^{-t} - X_3 \left(1 - e^{-t}\right) \,, \quad x_2 = X_2 - X_3 \left(e^{t} - e^{-t}\right) \,, \quad x_3 = X_3 e^{-t}$$

is the motion law. Find the velocity and acceleration fields both in material and in spatial descriptions.

PROBLEM 3.3: Given the velocity field of a continuum: $\mathbf{v} = \mathbf{x}/(1+t)$: prove that the motion law then takes the form $\mathbf{x} = \mathbf{X}\,(1+t)$. Determine the velocity and acceleration fields both in material (Lagrangian) and in spatial (Eulerian) descriptions.

[1] Osbrone Reynolds, 1842-1912

PROBLEM 3.4: Given the velocity field for a motion in the following form:

$$v_1 = \alpha x_3\,, \qquad v_2 = -\beta x_3\,, \qquad v_3 = -\alpha x_1 + \beta x_2\,,$$

where α and β are non zero constants. Verify that this motion is a rigid body motion. Find the spin vector.

PROBLEM 3.5: Given the velocity field of a continuum in spatial description:

$$v_1 = -\frac{2x_1x_2x_3}{R^4}\,, \quad v_2 = \frac{x_1^2 - x_2^2}{R^4}x_1\,, \quad v_3 = \frac{x_2}{R^2}\,,$$

where $R = \sqrt{x_1^2 + x_2^2} \neq 0$. Find the velocity gradient, the strain rate tensor, the spin tensor, the vorticity vector and the acceleration.

PROBLEM 3.6: Assume that the velocity field is the gradient of a potential function ϕ, i.e., $\mathbf{v} = \phi\nabla$. Prove that the right side of equation

$$\frac{\mathrm{D}}{\mathrm{D}t}(\mathbf{v}) = (\mathbf{v})^{\boldsymbol{\cdot}} = \left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\nabla$$

is really the acceleration field.

PROBLEM 3.7: Given the velocity field of a continuum in spatial description:

$$v_1 = \frac{f(R)}{R}x_2\,, \quad v_2 = -\frac{f(R)}{R}x_1\,, \quad v_3 = 0\,,$$

where $R = \sqrt{x_1^2 + x_2^2} \neq 0$. Prove that this motion is volume preserving. Show, in addition to this, that the spin vector (or the angular velocity vector) vanishes if $f(R) = -1/R$.

PROBLEM 3.8: Show the validity of equation (3.59) in direct notation.

PROBLEM 3.9: Show the validity of equation (3.61) in indicial notation.

CHAPTER 4

Kinematic linearization

4.1. Linearization of the deformation gradients and strain tensors

There are many engineering problems for which the magnitudes of the axial strains ε^e and angle changes γ_{12} are, in general, much smaller than one:

$$|\varepsilon^e| \ll 1\,, \qquad\qquad |\gamma_{12}| \ll 1\,. \tag{4.1}$$

Then it follows from (2.44) and (2.59) that the components of the Green-Lagrange strain tensor $\boldsymbol{E}$ and the Euler-Almansi strain tensor $\boldsymbol{e}$ are also much smaller than one. If condition (4.1) is satisfied we speak about small deformations.

Fulfillment of condition (4.1) does, however, not necessarily mean that the magnitudes of the displacement gradient components $u_{A,B}$ and $u_{k,\ell}$ are much smaller than one. It may also occur that the rotations are finite though the strains and angle changes are small.

In the sequel we shall, in general, assume that the magnitudes of the displacement gradient components are much smaller than one, i.e., the inequalities

$$|u_{A,B}| \ll 1\,, \qquad\qquad |u_{k,\ell}| \ll 1\,. \tag{4.2}$$

are also satisfied. If this is the case the rotations are also small. Then (a) the quadratic terms in $u_{A,B}$, $u_{k,\ell}$ can be neglected and in addition to this (b) the linear terms in $u_{A,B}$, $u_{k,\ell}$ can also be neglected provided that they are compared to the unit.

If conditions (4.1) and (4.2) are all satisfied we speak about the linear theory of deformation in contrast to the theory of small deformations.

The deformation gradients can be given in terms of the displacement vectors:

$$\boldsymbol{F} = \boldsymbol{1} + \mathbf{u}^\circ \circ \nabla^\circ \qquad \text{and} \qquad \boldsymbol{F}^{-1} = \boldsymbol{1} - \mathbf{u} \circ \nabla\,. \tag{4.3}$$

According to equation $(2.30)_2$ it also holds that

$$\nabla = \nabla^\circ \cdot \boldsymbol{F}^{-1}\,. \tag{4.4}$$

from where substituting $(4.3)_2$ we have

$$\boxed{\nabla = \nabla^\circ \cdot \boldsymbol{F}^{-1} = \nabla^\circ \cdot (\boldsymbol{1} - \mathbf{u} \circ \nabla)} \tag{4.5a}$$

or

$$\nabla_a = \nabla^\circ_B \left(\delta_{ba} - u_b \nabla_a\right) = \nabla^\circ_B \left(\delta_{ba} - u_{b,a}\right) \;, \qquad a = A,\;\; b = B \tag{4.5b}$$

in which $|u_{b,a}| \ll 1$. Hence

$$\nabla_a \cong \nabla^\circ_A \;, \qquad a = A; \qquad \nabla \cong \nabla^\circ \tag{4.6}$$

by the use of which we have

$$\mathbf{u} \circ \nabla = \mathbf{u}^\circ \circ \nabla^\circ \tag{4.7}$$

and

$$\boldsymbol{F} = \boldsymbol{1} + \mathbf{u} \circ \nabla \,, \qquad\qquad \boldsymbol{F}^{-1} = \boldsymbol{1} - \mathbf{u} \circ \nabla \,. \tag{4.8}$$

A comparison of equations (2.39), (2.55) and (4.6) yields

$$\boxed{\boldsymbol{\varepsilon}^\circ = \boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u}\right) .} \tag{4.9}$$

REMARK 4.1: Note that here we have dropped the terms quadratic in $u_{A,B}$ and $u_{k,\ell}$. In addition we have utilized the notations introduced in Remarks 2.8 and 2.17. Since the Green-Lagrange and Euler-Almansi strain tensors coincide with each other we shall denote them by the same letter $\boldsymbol{\varepsilon}$ in the linear theory of deformations.

When writing linearized equations the superscript $^\circ$ will be, in general, dropped and we shall use small Latin letters (lowercase letters) for the subscripts. For instance:

$$\varepsilon_{k\ell} = \frac{1}{2}\left(u_{k,\ell} + u_{\ell,k}\right) \qquad \text{and} \qquad \underset{(3\times3)}{\underline{\varepsilon}} = [\varepsilon_{k\ell}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \tag{4.10}$$

are the infinitesimal strain tensor and its matrix. As regards the position vectors, and the letters used to identify the positions of the material points before and after deformation we shall keep, however, the earlier notation conventions – Figure 4.1 clearly represents these notation conventions.

With the infinitesimal strain tensor it follows from (2.38) (or(2.54)) – $\boldsymbol{\varepsilon}^\circ = \boldsymbol{\varepsilon}$ – that

$$\varepsilon_{kk} = \mathbf{i}_k \cdot \boldsymbol{\varepsilon} \cdot \mathbf{i}_k \qquad (\text{no sum on } k) \tag{4.11}$$

is the axial strain in the direction $\mathbf{i}_k$ at the point of the body where the strain tensor is considered.

Since $|\gamma_{12}| \ll 1$ and the axial strains are also much less than one equation (2.45) yields the angle change between the directions $\mathbf{i}_k$ and $\mathbf{i}_\ell$ in the following form:

$$\gamma_{k\ell} = 2\mathbf{i}_k \cdot \boldsymbol{\varepsilon} \cdot \mathbf{i}_\ell = 2\varepsilon_{k\ell} \,, \qquad k \neq \ell \,. \tag{4.12}$$

We remark that the same formula follows from equation (2.59).

With $(4.10)_1$, (4.11) and (4.12) we can rewrite matrix $(4.10)_2$:

$$[\varepsilon_{k\ell}] = \begin{bmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2}+u_{2,1}) & \frac{1}{2}(u_{1,3}+u_{3,1}) \\ \frac{1}{2}(u_{2,1}+u_{1,2}) & u_{2,2} & \frac{1}{2}(u_{2,3}+u_{3,2}) \\ \frac{1}{2}(u_{3,1}+u_{1,3}) & \frac{1}{2}(u_{3,2}+u_{2,3}) & u_{3,3} \end{bmatrix} . \quad (4.13)$$

4.2. Geometrical description of the linear deformations

We can now determine the right stretch tensor and its inverse. Upon substitution of (4.8) into (2.74) we obtain:

$$\boldsymbol{U} = \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{F}^T \cdot \boldsymbol{F}} = \sqrt{(\boldsymbol{1} + \nabla \circ \mathbf{u}) \cdot (\boldsymbol{1} + \mathbf{u} \circ \nabla)} \approx$$
$$\approx \sqrt{\boldsymbol{1} + \mathbf{u} \circ \nabla + \nabla \circ \mathbf{u}} = \sqrt{\boldsymbol{1} + 2\boldsymbol{\varepsilon}} \approx \boldsymbol{1} + \boldsymbol{\varepsilon} \quad (4.14a)$$

and

$$\boldsymbol{U}^{-1} = \sqrt{\boldsymbol{C}^{-1}} = \sqrt{\boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T}} = \sqrt{(\boldsymbol{1} - \mathbf{u} \circ \nabla) \cdot (\boldsymbol{1} - \nabla \circ \mathbf{u})} \approx$$
$$\approx \sqrt{\boldsymbol{1} - (\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u})} = \sqrt{\boldsymbol{1} - 2\boldsymbol{\varepsilon}} \approx \boldsymbol{1} - \boldsymbol{\varepsilon} . \quad (4.14b)$$

We proceed with the rotation tensor. Equation (2.76) yields

$$\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \underset{(4.8)_1 \ (4.14b)}{=\!\!\uparrow\!\!=} (\boldsymbol{1} + \mathbf{u} \circ \nabla) \cdot (\boldsymbol{1} - \boldsymbol{\varepsilon}) \approx$$
$$\approx \boldsymbol{1} + \mathbf{u} \circ \nabla - \boldsymbol{\varepsilon} = \boldsymbol{1} + \mathbf{u} \circ \nabla - \frac{1}{2}(\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u}) = \boldsymbol{1} + \frac{1}{2}(\mathbf{u} \circ \nabla - \nabla \circ \mathbf{u}) =$$
$$= \boldsymbol{1} + \boldsymbol{\Psi} , \quad (4.15)$$

where

$$\boxed{\boldsymbol{\Psi} = \frac{1}{2}(\mathbf{u} \circ \nabla - \nabla \circ \mathbf{u})} \quad (4.16)$$

is the tensor of infinitesimal rotation. Since the mapping

$$\boldsymbol{R} \cdot \mathrm{d}\mathbf{X} = (\boldsymbol{1} + \boldsymbol{\Psi}) \cdot \mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{X} + \boldsymbol{\Psi} \cdot \mathrm{d}\mathbf{X}$$

is distance preserving – $|\boldsymbol{R} \cdot \mathrm{d}\mathbf{X}| = |\mathrm{d}\mathbf{X}|$ – it follows that the product

$$\boldsymbol{\Psi} \cdot \mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{u}$$

is the displacement that belongs to the tip of the vector $\mathrm{d}\mathbf{X}$.

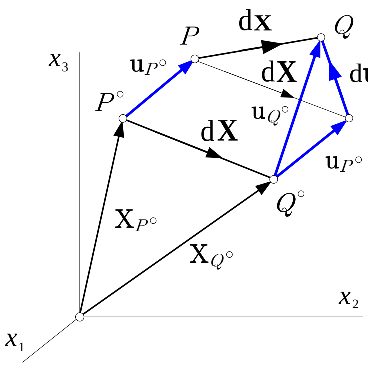

FIGURE 4.1. Displacements of two neighboring material points

Figure 4.1 shows the displacements of the material points P° and Q° under the assumption that both $\mathrm{d}\mathbf{X}$ and $\mathrm{d}\mathbf{x}$ are sufficiently small. It is obvious that

$$\mathbf{u}\circ\nabla = \underbrace{\frac{1}{2}\left(\mathbf{u}\circ\nabla + \nabla\circ\mathbf{u}\right)}_{\boldsymbol{\varepsilon}} + \underbrace{\frac{1}{2}\left(\mathbf{u}\circ\nabla - \nabla\circ\mathbf{u}\right)}_{\boldsymbol{\Psi}} = \boldsymbol{\varepsilon} + \boldsymbol{\Psi} \tag{4.17}$$

is the additive resolution of the displacement gradient. Utilizing equations (2.14) and $(4.8)_1$ we can write

$$\mathrm{d}\mathbf{x} = \boldsymbol{F}|_{P^\circ}\cdot\mathrm{d}\mathbf{X} = (\boldsymbol{1} + \mathbf{u}\circ\nabla)|_{P^\circ}\cdot\mathrm{d}\mathbf{X} = \mathrm{d}\mathbf{X} + \underbrace{(\mathbf{u}\circ\nabla)|_{P^\circ}\cdot\mathrm{d}\mathbf{X}}_{\mathrm{d}\mathbf{u}}\,, \tag{4.18}$$

where

$$\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{X} + \mathrm{d}\mathbf{u}\,, \qquad \mathrm{d}\mathbf{u} = \mathbf{u}|_{Q^\circ} - \mathbf{u}|_{P^\circ} \tag{4.19}$$

Substitute (4.19) into (4.18) and take the additive resolution (4.17) into account. We obtain

$$\mathbf{u}_{Q^\circ} = \underbrace{\mathbf{u}_{P^\circ}}_{\substack{\text{shift}\\ \text{(translation)}}} + \underbrace{(\mathbf{u}\circ\nabla)|_{P^\circ}\cdot\mathrm{d}\mathbf{X}}_{\text{relative displacement}} \tag{4.20a}$$

or

$$\mathbf{u}_{Q^\circ} = \underbrace{\underbrace{\mathbf{u}_{P^\circ}}_{\substack{\text{shift}\\ \text{(translation)}}} + \underbrace{\boldsymbol{\Psi}|_{P^\circ}\cdot\mathrm{d}\mathbf{X}}_{\substack{\text{displacement}\\ \text{from rotation}}}}_{\text{ridid body motion}} + \underbrace{\boldsymbol{\varepsilon}|_{P^\circ}\cdot\mathrm{d}\mathbf{X}}_{\substack{\text{displacement from}\\ \text{pure deformation}}} \,. \tag{4.20b}$$

The above equation is a geometrical interpretation of the motion in the neighborhood of the material point P°.

The tensor $\boldsymbol{\Psi}$ is skew. Assume that we know its axial vector $\boldsymbol{\Psi}^{(a)} = \boldsymbol{\varphi}$. Recalling equations (1.90) and $(1.91)_1$ we can calculate the displacement from rotation in the small neighborhood of any point within the body by using the formula

$$\boxed{\boldsymbol{\Psi}\cdot\mathrm{d}\mathbf{X} = \boldsymbol{\varphi}\times\mathrm{d}\mathbf{X} \qquad \text{where} \qquad \boldsymbol{\psi}^{(a)} = \boldsymbol{\varphi} = -\frac{1}{2}\,\mathbf{u}\times\nabla\,.} \tag{4.20c}$$

It also follows from equations (1.93) and $(1.91)_2$ that

$$\boxed{\Psi_{k\ell} = -e_{k\ell r}\varphi_r\,, \qquad \psi_r^{(a)} = \varphi_r = -\frac{1}{2}\,u_{p,q}e_{pqr}\,.} \tag{4.21a}$$

Hence

$$\begin{aligned} \Psi_{12} &= -\Psi_{21} = -e_{123}\varphi_3 = -\varphi_3\,, \\ \Psi_{23} &= -\Psi_{32} = -e_{231}\varphi_1 = -\varphi_1\,, \\ \Psi_{31} &= -\Psi_{13} = -e_{312}\varphi_2 = -\varphi_2\,, \\ &\Psi_{k\ell} = 0 \ \text{ if } \ k \neq \ell\,, \end{aligned} \qquad [\Psi_{k\ell}] = \begin{bmatrix} 0 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 0 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 0 \end{bmatrix}. \tag{4.21b}$$

REMARK 4.2: The axial vector (rotation vector) $\boldsymbol{\psi}^{(a)} = \boldsymbol{\varphi}$ describes small rotations.

It also holds that

$$\Psi_{k\ell} = -e_{k\ell r}\psi_r^{(a)} = -\delta_{km}e_{m\ell r}\psi_r^{(s)} = \delta_{km}\psi_r^{(a)}e_{m\ell r} \tag{4.22a}$$

which can be given in symbolic notation as

$$\boldsymbol{\Psi} = \boldsymbol{1} \times \boldsymbol{\psi}^{(a)}\,. \tag{4.22b}$$

4.3. Volume and surface elements

It follows from (2.20) and $(2.32)_1$ that

$$J = \det(\boldsymbol{F}) = \sqrt{\det(\boldsymbol{C})} = \det(\boldsymbol{U})\,. \tag{4.23}$$

If we know the principal values of the stretch ratios we can use $(2.75a)_2$ for determining the Jacobian:

$$J = \det(\boldsymbol{U}) = \lambda_1\lambda_2\lambda_3\,, \tag{4.24}$$

where, recalling equation, (2.36) we can give λ_ℓ in terms of the principal strains ε_ℓ: $\lambda_\ell = 1 + \varepsilon_\ell$. With these formulae the volume change (2.94) can be rewritten into the following form

$$\begin{aligned} \mathrm{d}V = J\mathrm{d}V^\circ = \lambda_1\lambda_2\lambda_3\mathrm{d}V^\circ = (1+\varepsilon_1)\,(1+\varepsilon_2)\,(1+\varepsilon_3)\,\mathrm{d}V^\circ \approx \\ \approx \mathrm{d}V^\circ + \underbrace{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)}_{\varepsilon_I}\mathrm{d}V^\circ \end{aligned} \tag{4.25}$$

in which ε_I is the first scalar invariant of the strain tensor $\boldsymbol{\varepsilon}$. The dilatation ε_V is defined by the equation

$$\varepsilon_V = \frac{\mathrm{d}V - \mathrm{d}V^\circ}{\mathrm{d}V^\circ} = \lambda_1\lambda_2\lambda_3 - 1 \approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_I\,. \tag{4.26}$$

With the dilatation

$$\Delta V = \int_{V^\circ} \varepsilon_V\,\mathrm{d}V^\circ \tag{4.27}$$

is the volume change of the body.

It is also clear that

$$J \approx 1 + \varepsilon_I \approx 1\,. \tag{4.28}$$

As regards the relations we have established between the vectorial and scalar surface elements we may write

$$\mathrm{d}\mathbf{A} = \underbrace{J}_{\approx 1+\varepsilon_I} \underbrace{\boldsymbol{F}^{-T}}_{\boldsymbol{1}-\nabla\circ\mathbf{u}\approx\boldsymbol{1}} \cdot \mathrm{d}\mathbf{A}^{\circ} \approx \mathrm{d}\mathbf{A}^{\circ} \tag{4.29a}$$

Hence it also holds that

$$\mathrm{d}A \approx \mathrm{d}A^{\circ}\,. \tag{4.29b}$$

EXERCISE 4.1: The displacement field in the unit cube of Exercise 2.1 in material description is given by solution (B.2.6) to Problem 2.3:

$$u_1 = a_1 X_2^2\,, \quad u_2 = a_2\,, \quad u_3 = a_3 X_2\, X_3\,.$$

Find the matrices of (a) the displacement gradient, (b) the infinitesimal strain tensor, (c) the infinitesimal rotation tensor and (d) determine the components of the infinitesimal rotation vector.

It can be checked with ease that

$$\left.\begin{aligned}
[u_{k,\ell}] &= \begin{bmatrix} 0 & 2a_1\,X_2 & 0 \\ 0 & 0 & 0 \\ 0 & a_3\,X_3 & a_3\,X_2 \end{bmatrix}, \\
[\varepsilon_{k\ell}] = \left[u_{(k,\ell)}\right] &= \begin{bmatrix} 0 & a_1\,X_2 & 0 \\ a_1\,X_2 & 0 & a_3\,X_3/2 \\ 0 & a_3\,X_3/2 & a_3\,X_2 \end{bmatrix}, \\
[\Psi_{k\ell}] = \left[u_{[k,\ell]}\right] &= \begin{bmatrix} 0 & a_1\,X_2 & 0 \\ -a_1\,X_2 & 0 & -a_3\,X_3/2 \\ 0 & a_3\,X_3/2 & 0 \end{bmatrix}.
\end{aligned}\right\} \tag{4.30}$$

Making use of equation (4.21b) we can now find the components of the infinitesimal rotation vector:

$$\varphi_1 = a_3 X_3/2\,, \qquad \varphi_2 = 0\,, \qquad \varphi_3 = -a_1 X_2\,.$$

If we drop the quadratic terms in solution (B.2.8) of Problem 2.5 we shall find that the Green-Lagrange stain tensor coincides with solution (4.30): $\boldsymbol{E} \approx \boldsymbol{\varepsilon}$.

4.4. Equations in cylindrical coordinate system

Equation

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{u}\circ\nabla + \nabla\circ\mathbf{u}\right), \tag{4.31}$$

which gives the strain tensor of the linear theory, is called either kinematic equation, or defining equation. The second name reflects the fact that the above equation defines the strain tensor in terms of the displacement vector. The later is called fundamental variable since the displacement field is, in general, the unknown we would like to determine. The terms defining equation, fundamental variable we have introduced are those of E. Tonti [57, 58, 51, 52] who has established a unified classification for the field equations of some problems in mathematical physics.

The kinematic equation can be given not only in a Cartesian coordinate system but in any curvilinear coordinate system. If the cylindrical coordinate system $(R\vartheta z)$ is our choice first we have to give the displacement vector in this coordinate system. Recalling (1.197) which is formally a velocity field given in cylindrical coordinates we get

$$\mathbf{u}(R,\vartheta,z) = u_R\,\mathbf{i}_R + u_\vartheta\,\mathbf{i}_\vartheta + u_z\,\mathbf{i}_z \tag{4.32}$$

in which u_R, u_ϑ and u_z are functions of R, ϑ and z. If we now utilize both (1.195) and (1.198) we arrive at the following result

$$\boldsymbol{\mathcal{U}} = \mathbf{u}\circ\nabla = \boldsymbol{u}_R\circ\mathbf{i}_R + \boldsymbol{u}_\vartheta\circ\mathbf{i}_\vartheta + \boldsymbol{u}_z\circ\mathbf{i}_z\,, \tag{4.33}$$

where

$$\boldsymbol{u}_R = \frac{\partial\mathbf{u}}{\partial R} = \frac{\partial u_R}{\partial R}\,\mathbf{i}_R + \frac{\partial u_\vartheta}{\partial R}\,\mathbf{i}_\vartheta + \frac{\partial u_z}{\partial R}\,\mathbf{i}_z\,, \tag{4.34a}$$

$$\boldsymbol{u}_\vartheta = \frac{1}{R}\frac{\partial\mathbf{u}}{\partial\vartheta} = \left(\frac{1}{R}\frac{\partial u_R}{\partial\vartheta} - \frac{u_\vartheta}{R}\right)\mathbf{i}_R + \left(\frac{1}{R}\frac{\partial u_\vartheta}{\partial\vartheta} + \frac{u_R}{R}\,\mathbf{i}_\vartheta\right) + \frac{\partial u_z}{\partial\vartheta}\,\mathbf{i}_R \tag{4.34b}$$

and

$$\boldsymbol{u}_z = \frac{\partial\mathbf{u}}{\partial z} = \frac{\partial u_R}{\partial z}\,\mathbf{i}_R + \frac{\partial u_\vartheta}{\partial z}\,\mathbf{i}_\vartheta + \frac{\partial u_z}{\partial z}\,\mathbf{i}_R \tag{4.34c}$$

are the images of the unit vectors $\mathbf{i}_R$, $\mathbf{i}_\vartheta$ and $\mathbf{i}_z$. With $\boldsymbol{u}_R$, $\boldsymbol{u}_\vartheta$ and $\boldsymbol{u}_z$

$$\underset{(3\times3)}{\underline{\boldsymbol{\mathcal{U}}}} = \left[\ \underset{(3\times1)}{\underline{\boldsymbol{u}_R}}\ \middle|\ \underset{(3\times1)}{\underline{\boldsymbol{u}_\vartheta}}\ \middle|\ \underset{(3\times1)}{\underline{\boldsymbol{u}_z}}\ \right] = \begin{bmatrix} \dfrac{\partial u_R}{\partial R} & \dfrac{1}{R}\dfrac{\partial u_R}{\partial\vartheta} - \dfrac{u_\vartheta}{R} & \dfrac{\partial u_R}{\partial z} \\ \dfrac{\partial u_\vartheta}{\partial R} & \dfrac{1}{R}\dfrac{\partial u_\vartheta}{\partial\vartheta} + \dfrac{u_R}{R} & \dfrac{\partial u_\vartheta}{\partial z} \\ \dfrac{\partial u_z}{\partial R} & \dfrac{\partial u_z}{\partial\vartheta} & \dfrac{\partial u_z}{\partial z} \end{bmatrix}. \tag{4.35}$$

is the matrix of the displacement gradient in cylindrical coordinates. By applying the additive resolution theorem we can now determine the linear strain tensor

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{u}\circ\nabla + \nabla\circ\mathbf{u}\right) \tag{4.36}$$

and its matrix

$$\underset{(3\times3)}{\underline{\boldsymbol{\varepsilon}}} = \left[\ \underset{(3\times1)}{\underline{\boldsymbol{\varepsilon_R}}}\ \middle|\ \underset{(3\times1)}{\underline{\boldsymbol{\varepsilon_\vartheta}}}\ \middle|\ \underset{(3\times1)}{\underline{\boldsymbol{\varepsilon_z}}}\ \right] = \begin{bmatrix} \varepsilon_{RR} & \varepsilon_{R\vartheta} & \varepsilon_{Rz} \\ \varepsilon_{\vartheta R} & \varepsilon_{\vartheta\vartheta} & \varepsilon_{\vartheta z} \\ \varepsilon_{zR} & \varepsilon_{\vartheta z} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{RR} & \frac{1}{2}\gamma_{R\vartheta} & \frac{1}{2}\gamma_{Rz} \\ \frac{1}{2}\gamma_{\vartheta R} & \varepsilon_{\vartheta\vartheta} & \frac{1}{2}\gamma_{\vartheta z} \\ \frac{1}{2}\gamma_{zR} & \frac{1}{2}\gamma_{\vartheta z} & \varepsilon_{zz} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial u_R}{\partial R} & \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \vartheta} - \frac{u_\vartheta}{R} + \frac{\partial u_\vartheta}{\partial R}\right) & \frac{1}{2}\left(\frac{\partial u_R}{\partial z} + \frac{\partial u_z}{\partial R}\right) \\ \frac{1}{2}\left(\frac{\partial u_\vartheta}{\partial R} + \frac{1}{R}\frac{\partial u_R}{\partial \vartheta} - \frac{u_\vartheta}{R}\right) & \frac{1}{R}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_R}{R} & \frac{1}{2}\left(\frac{\partial u_\vartheta}{\partial z} + \frac{\partial u_z}{\partial \vartheta}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial R} + \frac{\partial u_R}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial \vartheta} + \frac{\partial u_\vartheta}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix} . \tag{4.37}$$

EXERCISE 4.2: Assume that the displacement field is axisymmetric and the axis z is that of the symmetry. Then the displacement field is independent of ϑ and is of the form

$$\mathbf{u}(R,z) = u_R\,\mathbf{i}_R + u_z\,\mathbf{i}_z\,. \tag{4.38}$$

Determine the matrix of the linear strain tensor.

It follows from (4.37) that the matrix of the strain tensor is simplified to

$$\underset{(3\times3)}{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \frac{\partial u_R}{\partial R} & 0 & \frac{1}{2}\left(\frac{\partial u_R}{\partial z} + \frac{\partial u_z}{\partial R}\right) \\ 0 & \frac{u_R}{R} & \frac{1}{2}\frac{\partial u_z}{\partial \vartheta} \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial R} + \frac{\partial u_R}{\partial z}\right) & \frac{1}{2}\frac{\partial u_z}{\partial \vartheta} & \frac{\partial u_z}{\partial z} \end{bmatrix} . \tag{4.39}$$

4.5. Compatibility

For a given and differentiable displacement field

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{u}\circ\nabla + \nabla\circ\mathbf{u}\right)\,, \qquad \varepsilon_{k\ell} = \frac{1}{2}\left(u_{k,\ell} + u_{\ell,k}\right) \tag{4.40}$$

is the linearized (infinitesimal) strain tensor. If we take the symmetry of the strain tensor into account we come to the conclusion that the six tensor components $\varepsilon_{k\ell}$ are uniquely determined by tree displacement components u_ℓ. We can raise the opposite question: What if a strain tensor is given and the displacement field is unknown. Then, because of the symmetry of the strain tensor, we have six equations – see equations (4.40) – for three unknowns, i.e., the problem is overdetermined: we have more equations as there are unknowns. If for a given strain tensor $\boldsymbol{\varepsilon}$ there exists such a displacement field $\mathbf{u}$ that equation (4.40) is satisfied then the strains $\varepsilon_{k\ell}$ (the strain tensor $\boldsymbol{\varepsilon}$) are (is said to be) compatible.

Saint-Venant[1] was the first who attacked this problem in a lecture presented in Paris on July 28, 1860. His results were published later in 1861 [18, 25] though the correct proof of the results was given by Boussinesq[2] in 1871 [20].

Let us define the tensor of incompatibility $\boldsymbol{\eta}$ for a given strain tensor $\boldsymbol{\varepsilon}$ by the following relation

$$\boxed{\boldsymbol{\eta} = -\nabla \times \boldsymbol{\varepsilon} \times \nabla\,, \qquad \eta_{r\ell} = -e_{qpr}\nabla_q \varepsilon_{ps}\nabla_k e_{sk\ell} = e_{pqr}e_{sk\ell}\varepsilon_{ps,qk}\,.} \tag{4.41}$$

Note that $\boldsymbol{\eta}$ is a symmetric tensor.

REMARK 4.3: The concept of the incompatibility tensor was introduced by I. Kozák [62, 61]. In this respect it is also worth citing the following paper and thesis [63, 66] here.

The diagonal and off-diagonal components of the incompatibility tensor are given by the following equations:

$$\left.\begin{aligned} \eta_{11} &= \varepsilon_{22,33} + \varepsilon_{33,22} - \underbrace{2\varepsilon_{23,23}}_{\gamma_{23,23}}\,,\\ \eta_{22} &= \varepsilon_{33,11} + \varepsilon_{11,33} - \underbrace{2\varepsilon_{31,31}}_{\gamma_{31,31}}\,,\\ \eta_{33} &= \varepsilon_{11,22} + \varepsilon_{22,11} - \underbrace{2\varepsilon_{12,12}}_{\gamma_{12,12}} \end{aligned}\right\} \tag{4.42a}$$

and

$$\left.\begin{aligned} \eta_{12} &= \underbrace{\left(\varepsilon_{13,2} + \varepsilon_{23,1} - \varepsilon_{12,3}\right)_{,3}}_{\frac{1}{2}(\gamma_{13,2}+\gamma_{23,1}-\gamma_{12,3})_{,3}} - \varepsilon_{33,12}\,,\\ \eta_{23} &= \underbrace{\left(\varepsilon_{21,3} + \varepsilon_{31,2} - \varepsilon_{23,1}\right)_{,1}}_{\frac{1}{2}(\gamma_{21,3}+\gamma_{31,2}-\gamma_{23,1})_{,1}} - \varepsilon_{11,23}\,,\\ \eta_{31} &= \underbrace{\left(\varepsilon_{32,1} + \varepsilon_{12,3} - \varepsilon_{31,2}\right)_{,2}}_{\frac{1}{2}(\gamma_{32,1}+\gamma_{12,3}-\gamma_{31,2})_{,2}} - \varepsilon_{22,31}\,. \end{aligned}\right\} \tag{4.42b}$$

EXERCISE 4.3: Prove that

$$\begin{aligned} \boldsymbol{\eta} &= -\nabla \times \boldsymbol{\varepsilon} \times \nabla = \\ &= \left[(\operatorname{tr}\boldsymbol{\varepsilon})\nabla^2 - \nabla\cdot\boldsymbol{\varepsilon}\cdot\nabla\right]\boldsymbol{1} + \nabla\cdot\boldsymbol{\varepsilon}\circ\nabla + \nabla\circ\boldsymbol{\varepsilon}\cdot\nabla - \boldsymbol{\varepsilon}\nabla^2 - (\operatorname{tr}\boldsymbol{\varepsilon})\nabla\circ\nabla\,. \end{aligned} \tag{4.43}$$

In indicial notation

$$\eta_{r\ell} = e_{pqr}e_{sk\ell}\,\varepsilon_{ps,qk} \tag{4.44}$$

is the left side of the equation to be proved. Let us manipulate it into the desired form by (a) substituting (1.48) for the product of the two permutation symbols and

[1]Jean Claude Barré Saint-Venant, 1797-1886
[2]Joseph Valentin Boussinesq, 1842-1928

(b) expending then the determinant by the third row. If in addition to this we take into account that the Kronecker delta is an index renaming operator we get

$$\eta_{r\ell} = e_{pqr} e_{sk\ell}\, \varepsilon_{ps,qk} = \varepsilon_{ps,qk} \begin{vmatrix} \delta_{ps} & \delta_{pk} & \delta_{p\ell} \\ \delta_{qs} & \delta_{qk} & \delta_{q\ell} \\ \delta_{rs} & \delta_{rk} & \delta_{r\ell} \end{vmatrix} =$$

$$= \varepsilon_{pr,qk}\,(\delta_{pk}\delta_{q\ell} - \delta_{p\ell}\delta_{qk}) + \varepsilon_{ps,qr}\,(\delta_{p\ell}\delta_{qs} - \delta_{ps}\delta_{q\ell}) + \varepsilon_{ps,qk}\,(\delta_{ps}\delta_{qk} - \delta_{pk}\delta_{qs})\,\delta_{rl} =$$

$$= \underbrace{\varepsilon_{kr,lk} - \varepsilon_{rr,qq}}_{\nabla\cdot\boldsymbol{\varepsilon}\circ\nabla - \boldsymbol{\varepsilon}\boldsymbol{\Delta}} + \underbrace{\varepsilon_{lq,qr} - \varepsilon_{ss,lr}}_{\nabla\circ\boldsymbol{\varepsilon}\cdot\nabla - (\mathrm{tr}\,\boldsymbol{\varepsilon})\nabla\circ\nabla} + \underbrace{(\varepsilon_{pp,kk} - \varepsilon_{kq,kq})\,\delta_{rl}}_{(\mathrm{tr}\,\boldsymbol{\varepsilon})\Delta - \nabla\cdot\boldsymbol{\varepsilon}\cdot\nabla}\,. \tag{4.45}$$

That was to be proved.

Assume that $\boldsymbol{\varepsilon}$ is compatible, i.e., for the given strain tensor $\boldsymbol{\varepsilon}$ there exists such a displacement field which satisfies equation (4.40). Then

$$\boldsymbol{\eta} = -\nabla \times \boldsymbol{\varepsilon} \times \nabla = -\nabla \times \frac{1}{2}\,(\mathbf{u}\circ\nabla + \nabla\circ\mathbf{u}) \times \nabla =$$

$$= -\frac{1}{2}\,[\nabla \times (\mathbf{u}\circ\nabla) \times \nabla + \nabla \times (\nabla\circ\mathbf{u}) \times \nabla] = \boldsymbol{0}\,, \tag{4.46a}$$

or

$$\eta_{r\ell} = e_{pqr}\, e_{sk\ell}\, \varepsilon_{ps,qk} = \frac{1}{2}\, e_{pqr}\, e_{sk\ell}\, (u_{p,s} + u_{s,p})_{,qk} =$$

$$= \frac{1}{2}\, e_{pqr}\, e_{sk\ell}\, (u_{p,skq} + u_{s,pqk}) = 0\,. \tag{4.46b}$$

Consequently, the strain tensor is compatible if the tensor of incompatibility is zero tensor. The six equations

$$\boxed{\eta_{11} = \eta_{22} = \eta_{33} = 0\,, \quad \eta_{12} = \eta_{21} = 0\,, \quad \eta_{23} = \eta_{32} = 0\,, \quad \eta_{31} = \eta_{13} = 0} \tag{4.47}$$

the components of the compatible strain tensor $\boldsymbol{\varepsilon}$ should fulfill are called Saint-Venant's compatibility equations (conditions) [18, 25].

REMARK 4.4: Let $\boldsymbol{\varphi}(t)$ be an infinitesimal rotation vector. Further let $\mathbf{u}_P(t)$ be the displacement vector at the point P of the body $\boldsymbol{\mathcal{B}}$. The position vector of the point Q ($Q \neq P$) with respect to the point P is denoted by $\mathbf{x}_{PQ}(t)$. Equation

$$\mathbf{u} = \mathbf{u}_P + \boldsymbol{\varphi} \times \mathbf{x}_{PQ}(t)$$

describes a rigid body motion to which there belong no deformation, i.e., it holds that $\varepsilon_{ps} = 0$. This means that the equations of compatibility are identically satisfied for any rigid body motion. Consequently, the solution of equations (4.40) for the displacement field is not uniquely determined: it is determined with the accuracy of a rigid body motion only.

REMARK 4.5: If in a body every closed curve which does not intersect itself can be shrunken to a point in such a manner that the curve remains within the body then the body (the region occupied by the body) is said to by simply connected. In contrast to this if in a body there exist such closed curves which can not be shrunken to a point within the body then the body is said to by multiply connected.

Fulfillment of equations (4.46) is a necessary and sufficient condition for the strain field $\boldsymbol{\varepsilon}$ to be compatible if the body is simply connected. For multiply connected bodies, however, fulfillment of equations (4.46) is not sufficient for the strain field $\boldsymbol{\varepsilon}$ to be compatible: additional conditions should also be satisfied to ensure the compatibility of strains. We shall present these conditions in Subsection 4.6.

It follows from equation (4.41) that

$$\boldsymbol{\eta} \cdot \nabla = -(\nabla \times \boldsymbol{\varepsilon} \times \nabla) \cdot \nabla = \mathbf{0} \,. \tag{4.48}$$

Hence the six Saint Venant compatibility equations are not independent[3].

Fulfillment of equation (4.46) reduces the number of Saint-Venant's compatibility equation by three. It is, however, a further issue how to solve the following problems: (a) what are the three compatibility conditions to be satisfied (how to select them); (b) are there other conditions to ensure that the strains be compatible. In the sequel we shall investigate these problems on the basis of paper [62] and thesis [66].

Let $\boldsymbol{\alpha} = \alpha_{ab}(\mathbf{X})\, \mathbf{i}_a \circ \mathbf{i}_b$ be a sufficiently smooth and symmetric tensor field in V. Furthermore, let $\mathbf{w} = w_\ell(\mathbf{X})\, \mathbf{i}_\ell$ be an unknown vector field (a vector field to be determined later) in V. By $_{AB}$ we shall denote those subsets of the possible values of the index pairs $_{ab}$ for which the differential equations

$$\frac{1}{2}\left(w_{A,B} + w_{B,A}\right) = w_{(A,B)} = \alpha_{AB} \qquad \forall \mathbf{X} \in V \tag{4.49}$$

always have a solution for the vector field w_ℓ. It is obvious that the index pairs $_{AB}$ may have only three distinct values. For instance $_{11}$, $_{22}$, $_{33}$, or $_{12=21}$, $_{23=32}$, $_{31=13}$ (here, for symmetry reasons, the order of the indices does not count).

Let $_{RS}$ be the set of those index pairs the union of which with the set of index pairs $_{AB}$ is the set of index pairs $_{ab}$. (If $_{11}$, $_{22}$, $_{33}$ is the set of index pairs $_{AB}$ then $_{12=21}$, $_{23=32}$, $_{31=13}$ is the set of index pairs $_{RS}$.)

With regard to equation (4.48) it is obvious that

$$\mathbf{w} \cdot (\boldsymbol{\eta} \cdot \nabla) = \mathbf{w} \cdot \overset{\downarrow}{\boldsymbol{\eta}} \cdot \nabla = 0 \tag{4.50}$$

no matter what value $\boldsymbol{\varepsilon}$ has – the down arrow shows the quantity the operator ∇ is applied to. Integrate the above equation on the volume V of the body. If we apply the integration by parts theorem (1.180) we get

$$0 = \int_V \mathbf{w} \cdot \overset{\downarrow}{\boldsymbol{\eta}} \cdot \nabla \, \mathrm{d}V = \int_V (\mathbf{w} \cdot \boldsymbol{\eta}) \cdot \nabla \, \mathrm{d}V - \int_V \overset{\downarrow}{\mathbf{w}} \cdot \boldsymbol{\eta} \cdot \nabla \, \mathrm{d}V =$$
$$= \int_A \mathbf{w} \cdot \boldsymbol{\eta} \cdot \mathbf{n} \, \mathrm{d}A - \int_V \boldsymbol{\eta} \cdot\cdot (\mathbf{w} \circ \nabla) \, \mathrm{d}V \,. \tag{4.51}$$

In principle we can prescribe the fulfilment of condition $\boldsymbol{\eta} \cdot \mathbf{n} = 0$ on the surface A of the body. We shall call it compatibility boundary condition. (a) Assume that the compatibility boundary condition is fulfilled. (b) Since $\boldsymbol{\eta}$ is a symmetric tensor it also holds that

[3] The above equation corresponds to the Bianchi identity of the Riemann geometry [48]

$$\boldsymbol{\eta}\cdot\cdot(\mathbf{w}\circ\nabla)=\boldsymbol{\eta}\cdot\cdot\,(\mathbf{w}\circ\nabla)_{\text{sym}}+\underbrace{\boldsymbol{\eta}\cdot\cdot\,(\mathbf{w}\circ\nabla)_{\text{skew}}}_{=0}=$$
$$=\boldsymbol{\eta}\cdot\cdot\,(\mathbf{w}\circ\nabla)_{\text{sym}}=\eta_{ab}w_{(a,b)}\,. \tag{4.52}$$

By taking (a) and (b) into account we can rewrite equation (4.51) into the following form:

$$0=\underbrace{\int_A \mathbf{w}\cdot\boldsymbol{\eta}\cdot\mathbf{n}\,\mathrm{d}A}_{=0}-\int_V \boldsymbol{\eta}\cdot\cdot(\mathbf{w}\circ\nabla)\;\mathrm{d}V=\int_V \boldsymbol{\eta}\cdot\cdot(\mathbf{w}\circ\nabla)\;\mathrm{d}V=$$
$$=\int_V \eta_{ab}w_{(a,b)}\,\mathrm{d}V\,. \tag{4.53}$$

Consider now the equations

$$w_{(A,B)}=\alpha_{AB}=\eta_{AB}\,. \tag{4.54}$$

in which $w_\ell(\mathbf{X})$ is the unknown, and α_{AB} stands for η_{AB}.

Note that these equations are the same as equations (4.49).

Given the solution for w_ℓ, we can rewrite equation (4.53) into the form

$$\int_V\left[\sum_{AB}(\alpha_{AB})^2+\sum_{RS}w_{(R,S)}\eta_{RS}\right]dV=0 \tag{4.55}$$

in which the summation is to be carried out for every possible value of the index pairs $_{AB}$ and $_{RS}$. Assume that the equations of compatibility

$$\eta_{RS}=0 \tag{4.56}$$

are fulfilled. Then it follows from (4.55) that

$$\int_V\sum_{AB}(\alpha_{AB})^2dV=\int_V\sum_{AB}(\eta_{AB})^2]dV=0\,.$$

The above equation can be satisfied if and only if

$$\eta_{AB}=0 \qquad\qquad \forall\mathbf{X}\in V\,. \tag{4.57}$$

In other words the fulfillment of three compatibility conditions and the compatibility boundary conditions

$$\boxed{\eta_{RS}=0 \qquad \forall\mathbf{X}\in V \qquad \text{and} \qquad n_a\eta_{ab}=0 \qquad \forall\mathbf{X}\in A} \tag{4.58}$$

ensures the fulfillment of the equations

$$\eta_{AB}=0 \qquad\qquad \forall\mathbf{X}\in V\,. \tag{4.59}$$

Consequently, three field equations – see $(4.58)_1$ – and three boundary conditions – see $(4.58)_2$ – are equivalent to the six Saint-Venant's equations of compatibility.

4.6. Cesaro's formulae

Derivation of the Cesaro-Volterra formulae [33, 34] needs some preparations. We shall use, among others, the following relationship:

$$\boldsymbol{\psi}^{(a)} \circ \nabla = \nabla \times \boldsymbol{\varepsilon}\,. \tag{4.60a}$$

We can verify it if we substitute $(4.40)_1$ for $\boldsymbol{\varepsilon}$ and take formula (4.21a) for $\boldsymbol{\psi}^{(a)}$ into account:

$$\nabla \times \boldsymbol{\varepsilon} = \nabla \times \frac{1}{2}\left(\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u}\right) = -\frac{1}{2}\left(\mathbf{u} \times \nabla\right) \circ \nabla = \boldsymbol{\psi}^{(a)} \circ \nabla\,.$$

The product $\boldsymbol{\Psi} \circ \nabla$ can be transformed into a more suitable form if first we substitute (4.22b) and then make use of (4.60a). We get

$$\boldsymbol{\Psi} \circ \nabla = \left(\boldsymbol{1} \times \boldsymbol{\psi}^{(a)}\right) \circ \nabla = \boldsymbol{1} \times \left(\boldsymbol{\psi}^{(a)} \circ \nabla\right) = \boldsymbol{1} \times \left(\nabla \times \boldsymbol{\varepsilon}\right)\,. \tag{4.60b}$$

We have now all the tools we shall need to manipulate the expression $(\mathbf{u} \circ \nabla) \circ \nabla$ into such a form which makes possible to establish Cezaro's formulae. Substitute first the additive resolution (4.17) of the displacement gradient and take then into account the previous equation. We obtain

$$(\mathbf{u} \circ \nabla) \circ \nabla = \boldsymbol{\varepsilon} \circ \nabla + \boldsymbol{\Psi} \circ \nabla = \boldsymbol{\varepsilon} \circ \nabla + \boldsymbol{1} \times \left(\nabla \times \boldsymbol{\varepsilon}\right)\,. \tag{4.60c}$$

Making use of relations (4.60b) and (4.60c) we can determine the rotation vector $\boldsymbol{\psi}^{(a)} = \boldsymbol{\varphi}$ and the displacement vector $\mathbf{u}$ at any point of the body provided that we know the strain field $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(X_1, X_2, X_3)$ as well as the rotation vector $\boldsymbol{\varphi}_B$ and the displacement vector $\mathbf{u}_B$ at an arbitrary but fixed point (denoted by B) of the body – the vectors $\boldsymbol{\varphi}_B$ and $\mathbf{u}_B$ determine a rigid body motion of the body.

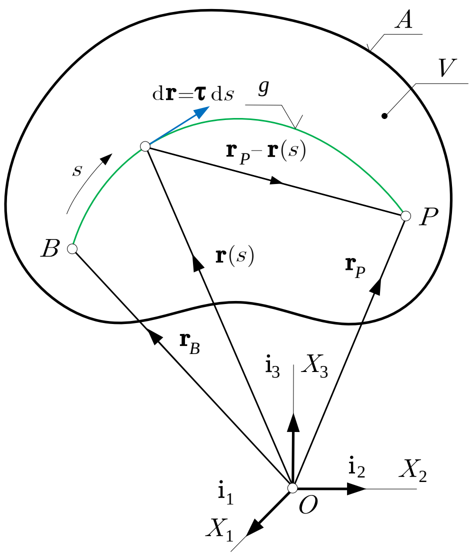

FIGURE 4.2. The material line g between the points B and P

Let us assume that $\mathbf{r}(s)$ is the equation of an arbitrary and differentiable space curve g which lies in the volume region V occupied by the body – see Figure 4.2 for details. The parameter s is, in general, the arc coordinate measured along the curve g. The curve begins at the point B while its endpoint is denoted by P. On the curve

$$\mathrm{d}\mathbf{r} = \boldsymbol{\tau}\,\mathrm{d}s\,, \qquad \boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \tag{4.61}$$

is the tangent vector and it follows from equations (4.60a) and (4.17) that

$$\mathrm{d}\boldsymbol{\varphi} = (\boldsymbol{\varphi} \circ \nabla) \cdot \mathrm{d}\mathbf{r} = (\nabla \times \boldsymbol{\varepsilon}) \cdot \mathrm{d}\mathbf{r} \tag{4.62}$$

and

$$\mathrm{d}\mathbf{u} = (\mathbf{u} \circ \nabla) \cdot \mathrm{d}\mathbf{r} = \boldsymbol{\varepsilon} \cdot \mathrm{d}\mathbf{r} + \boldsymbol{\Psi} \cdot \mathrm{d}\mathbf{r} = \boldsymbol{\varepsilon} \cdot \mathrm{d}\mathbf{r} + \boldsymbol{\varphi} \times \mathrm{d}\mathbf{r}\,. \tag{4.63}$$

If we integrate equation (4.62) along the curve g we obtain the rotation at P provided that we know $\boldsymbol{\varphi}_B$:

$$\boxed{\boldsymbol{\varphi}_P = \boldsymbol{\varphi}_B + \int_g (\nabla \times \boldsymbol{\varepsilon}) \cdot \mathrm{d}\mathbf{r}\,.} \tag{4.64}$$

In order to find the displacement vector $\mathbf{u}_P$ at P we have to integrate relation (4.63) along the curve g. For this purpose substitute the relation

$$\mathrm{d}\,\mathbf{r} = \mathrm{d}\,[\mathbf{r}_P - \mathbf{r}(s)] \tag{4.65}$$

into the second term on the right side of equation (4.63) and perform then an appropriate transformation:

$$\mathrm{d}\mathbf{u} = \boldsymbol{\varepsilon} \cdot \mathrm{d}\mathbf{r} + \mathrm{d}\,\{\boldsymbol{\varphi} \times [\mathbf{r}_P - \mathbf{r}(s)]\} - \mathrm{d}\boldsymbol{\varphi} \times [\mathbf{r}_P - \mathbf{r}(s)]\,. \tag{4.66}$$

We can now insert the expression $\mathrm{d}\boldsymbol{\psi}$ from equation (4.62) here. If integrate the result along the curve g – from B to P – we get the displacement vector at P:

$$\boxed{\mathbf{u}_P = \mathbf{u}_B + \boldsymbol{\varphi}_B \times [\mathbf{r}_P - \mathbf{r}_B] + \int_g \{\boldsymbol{\varepsilon} - [\mathbf{r}_P - \mathbf{r}(s)] \times (\nabla \times \boldsymbol{\varepsilon})\} \cdot \mathrm{d}\mathbf{r}\,.} \tag{4.67}$$

Equations (4.64) and (4.67) are the well known Cezaro-Volterra formulae.

Using indicial notation we can write $\mathrm{d}X_\ell = \tau_\ell\,\mathrm{d}s$ and $\nabla_m e_{mkr}\varepsilon_{k\ell}\tau_\ell \mathrm{d}s = e_{mkr}\varepsilon_{k\ell,m}\tau_\ell\,\mathrm{d}s$ for the terms $\mathrm{d}\mathbf{r}$ and $(\nabla \times \boldsymbol{\varepsilon}) \cdot \mathrm{d}\mathbf{r}$ in equation (4.64). Hence

$$\boxed{\varphi_r\,(P) = \varphi_r\,(B) + \int_g e_{mkr}\varepsilon_{k\ell,m}\tau_\ell\,\mathrm{d}s} \tag{4.68}$$

is the first Cezaro formula in indicial notation. It can be shown in the same manner that

$$\boxed{\begin{aligned} u_n(P) = u_n(B) + &e_{rqn}\varphi_r\,(B)\,(X_q\,(P) - X_q(B)) + \\ &+ \int_g \{\varepsilon_{n\ell} + [X_q\,(P) - X_q]\,e_{nqr}e_{mkr}\,\varepsilon_{k\ell,m}\}\,\tau_\ell\,\mathrm{d}s \end{aligned}} \tag{4.69}$$

is the second Cesaro formula in indicial notation.

REMARK 4.6: E. Cesaro and V. Volterra published these two relations in the first decade of the twentieth century [33, 34]. They assumed that the tensor ε was sufficiently smooth – derivable continuously at least twice. If ε is a square-integrable function their proof is not valid. For this case Ciarlet and his coauthors proved the two formulae in 2009 [88].

EXERCISE 4.4: Let N, ν, E and A be constants. Assume that the strain components are as follows:

$$\varepsilon_{11} = \frac{N}{AE}, \quad \varepsilon_{22} = \varepsilon_{33} = -\nu\frac{N}{AE},$$
$$\varepsilon_{12} = \varepsilon_{21} = \varepsilon_{23} = \varepsilon_{32} = \varepsilon_{31} = \varepsilon_{13} = 0\,.$$

Find the displacement field if there are no displacements and rotation at the origin.

The solution is given by the second Cesaro formula (4.69) in which B is the origin O and

$$u_n(O) = 0\,, \quad \varphi_r\,(O) = 0\,, \quad \varepsilon_{k\ell,m} = 0\,.$$

Hence

$$u_n(P) = u_n(B) + e_{rqn}\varphi_r\,(B)\,(X_q\,(P) - X_q(B)) + \int_g \{\varepsilon_{n\ell} + (X_q\,(P) - X_q(B))\,e_{nqr}e_{mkr}\,\varepsilon_{k\ell,m}\}\,\tau_\ell\,\mathrm{d}s = \int_g \varepsilon_{n\ell}\tau_\ell\,\mathrm{d}s\,.$$

The curve g is the union of three line segments g_1, g_2 and g_3 where: (a) g_1 is the line segment between the points $(0,0,0)$ and $(X_1,0,0)$, (b) g_2 is the line segment between the points $(X_1,0,0)$ and $(X_1,X_2,0)$, (c) g_3 is the line segment between the points $(X_1,X_2,0)$ and (X_1,X_2,X_3). See Figure 4.3 for details. The line segment

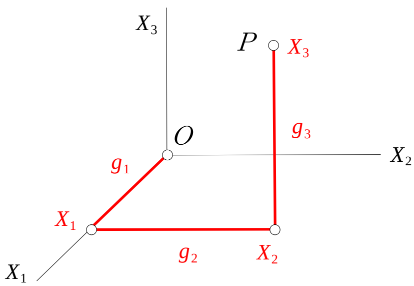

FIGURE 4.3. The union of the three line segments g_1, g_2 and g_3

$(0,0,0)$, $(X_1,0,0)$ lies on the axis X_1, the line segment $(X_1,0,0)$, $(X_1,X_2,0)$ is parallel to the axis X_2 and the line segment $(X_1,X_2,0)$ (X_1,X_2,X_3) is parallel to the axis X_3. Consequently, $\tau_\ell\,\mathrm{d}s$ is equal to

(a) $\tau_1\,\mathrm{d}s = \mathrm{d}X_1$ on the line segment $(0,0,0)$, $(X_1,0,0)$
(b) $\tau_2\,\mathrm{d}s = \mathrm{d}X_2$ on the line segment $(X_1,0,0)$, $(X_1,X_2,0)$ and
(b) $\tau_3\,\mathrm{d}s = \mathrm{d}X_3$ on the line segment $(X_1,X_2,0)$ (X_1,X_2,X_3).

On the basis of all that has been said above we get

$$u_1(P) = \int_0^{X_1} \varepsilon_{11} \underbrace{\tau_1 \, \mathrm{d}s}_{\mathrm{d}X_1} = \varepsilon_{11} \int_0^{X_1} \mathrm{d}X_1 = \varepsilon_{11} X_1,$$

$$u_2(P) = \int_0^{X_2} \varepsilon_{22} \underbrace{\tau_2 \, \mathrm{d}s}_{\mathrm{d}X_2} = \varepsilon_{22} \int_0^{X_2} \mathrm{d}X_2 = \varepsilon_{22} X_2,$$

$$u_3(P) = \int_0^{X_3} \varepsilon_{33} \underbrace{\tau_3 \, \mathrm{d}s}_{\mathrm{d}X_3} = \varepsilon_{33} \int_0^{X_2} \mathrm{d}X_3 = \varepsilon_{33} X_3.$$

Thus

$$u_1 = \frac{N}{AE} X_1, \qquad u_2 = -\nu \frac{N}{AE} X_2, \qquad u_3 = -\nu \frac{N}{AE} X_3 \,. \tag{4.70}$$

Let us assume that we have a bar with constant cross section A. Assume further that the bar is made of homogeneous and isotropic material for which E is Young's modulus and ν is Poisson's number. If the bar is subjected to an axial force N then solution (4.70) is the displacement field within the bar provided that the center line of the bar coincides with the axis X_1 and the origin is located at the left end of the bar.

4.7. Equations of compatibility from conditions of single valuedness

Let g be a piecewise continuous and closed curve in V which does not intersect itself. Furthermore let S be a surface, also in V, which is bounded by the curve g. The unit normal to S is $\mathbf{n}$, the unit tangent to g is $\boldsymbol{\tau}$, the binormal on g is denoted by $\boldsymbol{\nu}$ – see Figure 4.4, which shows g, S, $\boldsymbol{\tau}$, $\mathbf{n}$ and $\boldsymbol{\nu}$, for details.

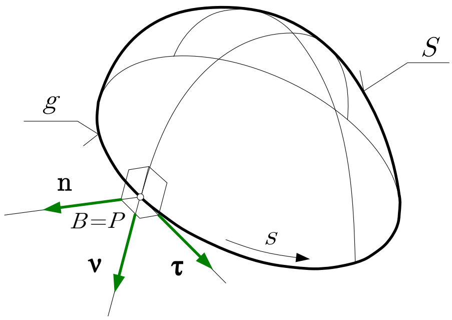

FIGURE 4.4. An open inner surface in V

Let the rotation and displacement fields be single valued. For $B = P$, i.e., for the closed curve g it follows from the Cezaro-Volterra formulae (4.64) and

(4.67) by taking the continuity conditions $\mathbf{u}_B = \mathbf{u}_P$, $\boldsymbol{\varphi}_B = \boldsymbol{\varphi}_P$ into account that

$$\boxed{\oint_g (\nabla \times \boldsymbol{\varepsilon}) \cdot \boldsymbol{\tau} \, \mathrm{d}s = \mathbf{0} \,, \qquad \oint_g \{\boldsymbol{\varepsilon} - [\mathbf{r}_P - \mathbf{r}(s)] \times (\nabla \times \boldsymbol{\varepsilon})\} \cdot \boldsymbol{\tau} \, \mathrm{d}s = \mathbf{0} \,.} \tag{4.71}$$

Equations (4.71) are called conditions of single valuedness or macro conditions of compatibly, or conditions of compatibility in the large.

If we make use of the Stokes theorem (1.181) – the cross product $(\nabla \times \boldsymbol{\varepsilon})$ corresponds to $\boldsymbol{H}$ in (1.181) – we can transform the first line integral in (4.71) into a surface integral:

$$\oint_g (\nabla \times \boldsymbol{\varepsilon}) \cdot \boldsymbol{\tau} \, \mathrm{d}s = \int_S [(\nabla \times \boldsymbol{\varepsilon}) \times \nabla] \cdot \mathbf{n} \, \mathrm{d}S = \mathbf{0} \,. \tag{4.72}$$

For any S bounded by g the previous integral can vanish if and only if

$$(\nabla \times \boldsymbol{\varepsilon}) \times \nabla = \nabla \times \boldsymbol{\varepsilon} \times \nabla = \underset{(4.46\mathrm{a})}{\uparrow} = -\boldsymbol{\eta} = \boldsymbol{O} \,. \tag{4.73}$$

Repeating the previous line of thought for the second line integral in (4.71) yields

$$\oint_g \{\boldsymbol{\varepsilon} - [\mathbf{r}_P - \mathbf{r}(s)] \times (\nabla \times \boldsymbol{\varepsilon})\} \cdot \boldsymbol{\tau} \, \mathrm{d}s =$$
$$= \int_S \Big\{ \{\boldsymbol{\varepsilon} - [\mathbf{r}_P - \mathbf{r}] \times (\nabla \times \boldsymbol{\varepsilon})\} \times \nabla \Big\} \cdot \mathbf{n} \, \mathrm{d}S = \mathbf{0} \,.$$

Hence

$$\boldsymbol{\varepsilon} \times \nabla - [(\mathbf{r}_P - \mathbf{r}) \times (\nabla \times \boldsymbol{\varepsilon})] \times \nabla = \boldsymbol{O} \,. \tag{4.74}$$

Consider now the second term on the right side. We can write

$$-[(\mathbf{r}_P - \mathbf{r}) \times (\nabla \times \boldsymbol{\varepsilon})] \times \nabla =$$
$$= -\underbrace{\left[(\mathbf{r}_P - \mathbf{r}) \frac{\partial}{\partial X_s} \right]}_{\mathbf{i}_s} \times (\nabla \times \boldsymbol{\varepsilon}) \times \mathbf{i}_s - (\mathbf{r}_P - \mathbf{r}) \times [(\nabla \times \boldsymbol{\varepsilon}) \times \nabla] \,, \tag{4.75}$$

where

$$\mathbf{i}_s \times (\nabla \times \boldsymbol{\varepsilon}) \times \mathbf{i}_s = \underset{\boldsymbol{\varepsilon} = \varepsilon_{k\ell} \mathbf{i}_k \circ \mathbf{i}_\ell}{\uparrow} = \mathbf{i}_s \times (\mathbf{i}_m \times \mathbf{i}_k) (\varepsilon_{k\ell} \nabla_m) \circ \mathbf{i}_\ell \times \mathbf{i}_s = \underset{(1.16),(1.39)}{\uparrow} =$$
$$= (\delta_{sk} \mathbf{i}_m - \delta_{sm} \mathbf{i}_k) (\varepsilon_{k\ell} \nabla_m) \circ \underbrace{\mathbf{i}_\ell \times \mathbf{i}_s}_{e_{\ell s p} \mathbf{i}_p} = \underbrace{e_{\ell s p} \varepsilon_{s\ell}}_{=0} \nabla_m \, \mathbf{i}_m \circ \mathbf{i}_p - \varepsilon_{k\ell} \, \mathbf{i}_k \circ \mathbf{i}_\ell \times \nabla_s \mathbf{i}_s = -\boldsymbol{\varepsilon} \times \nabla \,.$$

Consequently,

$$-[(\mathbf{r}_P - \mathbf{r}) \times (\nabla \times \boldsymbol{\varepsilon})] \times \nabla = -\boldsymbol{\varepsilon} \times \nabla - (\mathbf{r}_P - \mathbf{r}) \times [(\nabla \times \boldsymbol{\varepsilon}) \times \nabla] \,. \tag{4.76}$$

Upon substitution of this result into (4.74) we get

$$(\mathbf{r}_P - \mathbf{r}) \times [(\nabla \times \boldsymbol{\varepsilon}) \times \nabla] = \boldsymbol{O}$$

which holds for any $\mathbf{r}_P - \mathbf{r}$. Therefore the condition of single valuedness $(4.71)_2$ is equivalent to the compatibility equation

$$\boldsymbol{\eta} = -\nabla \times \boldsymbol{\varepsilon} \times \nabla = \boldsymbol{O} \,. \tag{4.77}$$

4.8. An overview of the results

As regards the conditions of single valuedness the most important results are as follows:

- Since the line integrals in (4.71) vanish independently of the fact what shape the closed curve g has it follows that the Cezaro-Voltera formulae (4.64) and (4.67) give the rotation vector $\boldsymbol{\varphi}_P$ and the displacement vector $\mathbf{u}_P$ independently of the shape the open curve g between the points B and P has – see Figure 4.2 – provided that the compatibility equations

$$\boldsymbol{\eta} = -(\nabla \times \boldsymbol{\varepsilon}) \times \nabla = \boldsymbol{O}\,, \quad \eta_{ab} = e_{akm} e_{b\ell p} \varepsilon_{k\ell,mp} = 0 \qquad \forall X_\ell \in V \tag{4.78}$$

  are satisfied in the simply connected volume region V. The vectors $\boldsymbol{\varphi}_B$ and $\mathbf{u}_B$ in equations (4.64) and (4.67) determine a rigid body motion of the body described by the relationship

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}_B + \boldsymbol{\psi}_B \times (\mathbf{r} - \mathbf{r}_B) \,.$$

  Consequently, as has already been mentioned in Remark 4.4, it is infinite the number of those rotation and displacement fields which belong to the single valued strain field $\boldsymbol{\varepsilon}(x^1, x^2, x^3)$ provided that the later satisfies the compatibility condition (4.78). These rotation and displacement fields may, however, differ from each other in a rigid body motion only.

  In other words: for a single valued strain field $\boldsymbol{\varepsilon}(x^1, x^2, x^3)$ and given $\boldsymbol{\psi}_B$ and $\mathbf{u}_B$ the rotation and displacement fields obtained form (4.64) and (4.65) are also single valued.
- For a simply connected region V vanishing of the line integrals in (4.71) is equivalent to the compatibility conditions (4.77) and conversely fulfillment of the compatibility conditions (4.77) ensures that the line integrals in (4.71) vanish.
- A torus is a double connected volume region V. Let g be a closed space curve inside the torus. We assume that g surrounds the hole in the torus and does not intersect itself. It is obvious that g can not be the boundary curve for such a simply-connected surface S which lies inside the torus. Consequently, we can not apply the line of thought for which equations (4.71) are the points of departure and which are based on the use of the Stokes theorem and resulted in the compatibility conditions (4.73) (or (4.77)). Hence for the strains to be compatible within the torus not only the mentioned compatibility conditions should be fulfilled

but the conditions of single valuedness (the compatibility conditions in the large) (4.71) as well on any such space curve g which surrounds the hole inside the torus. It can be proved – see for example [81] or [91] – that fulfillment of the compatibility conditions in the large on a space curve g which surrounds the hole of the torus ensures the fulfillment of these conditions on any other space curve which surrounds the hole.

4.9. Problems

PROBLEM 4.1: Given the displacement field of a solid body:

$$\boldsymbol{u} = C\left(X_1 X_2^2 \mathbf{i}_1 + X_2 X_3^2 \mathbf{i}_2 + X_3 X_1^2 \mathbf{i}_3\right); \qquad C = 10^{-3}\ \left[1/\mathrm{mm}^2\right]. \tag{4.79}$$

Show that

$$[\varepsilon_{k\ell}] = C\begin{bmatrix} X_2^2 & X_1 X_2 & X_1 X_3 \\ X_1 X_2 & X_3^2 & X_2 X_3 \\ X_1 X_3 & X_2 X_3 & X_1^2 \end{bmatrix}, \quad [\Psi_{k\ell}] = C\begin{bmatrix} 0 & X_1 X_2 & -X_1 X_3 \\ -X_1 X_2 & 0 & X_2 X_3 \\ X_1 X_3 & -X_2 X_3 & 0 \end{bmatrix}. \tag{4.80}$$

Determine the displacements at the points

$$\mathbf{X}_P = -20\mathbf{i}_1 + 30\mathbf{i}_2 + 40\mathbf{i}_3 \quad [\mathrm{mm}]\,, \qquad \mathbf{X}_Q = \mathbf{X}_P + \mathbf{i}_1 \quad [\mathrm{mm}]\,.$$

Then compare the value $\Delta\mathbf{u} = \mathbf{u}_Q - \mathbf{u}_P$ calculated using the exact solution for $\mathbf{u}$ and the approximation

$$\Delta\mathbf{u} \approx (\mathbf{u} \circ \nabla)|_{P^\circ} \cdot \Delta\mathbf{X}\,, \qquad \Delta\mathbf{X} = \mathbf{X}_Q - \mathbf{X}_P = \mathbf{i}_1 \quad [\mathrm{mm}]\,.$$

PROBLEM 4.2: Show that the diagonal and off-diagonal components of the incompatibility tensor are given by equations (4.42).

PROBLEM 4.3: Assume that (a) $\varepsilon_{\kappa\lambda}$ is independent of X_3 and $\varepsilon_{\kappa 3} = \varepsilon_{3\kappa} = \varepsilon_{33} = 0$. Find the compatibility equations for this strain tensor and clarify the conditions under which the strain components

$$\varepsilon_{11} = k(X_1^2 - X_2^2)\,, \qquad \varepsilon_{12} = \varepsilon_{21} = \ell X_1 X_2\,, \qquad \varepsilon_{22} = k X_1 X_2\,, \\ \varepsilon_{\kappa 3} = \varepsilon_{3\kappa} = \varepsilon_{33} = 0 \tag{4.81}$$

are compatible if k and ℓ are arbitrary not zero constants.

PROBLEM 4.4: The strain field in Problem 4.3 is compatible if $k = -\ell$. Find the rotation field.

PROBLEM 4.5: Find the displacement components for the strain field in Problem 4.3.

CHAPTER 5

Various stress measures

5.1. The stress vector concept

5.1.1. Interaction on inner surfaces of the body. Consider a material body in the current configuration. The forces exerted on the body (or within the body) can be divided into two groups depending on what the causes of the forces are: if they are exerted on the body by other bodies we speak about external forces, however, if we consider the forces exerted by one part of the body on another part of the body we speak about inner forces.

The external forces can again be separated into two distinct classes: body forces which act on the volume (or mass) elements of the body, and surface

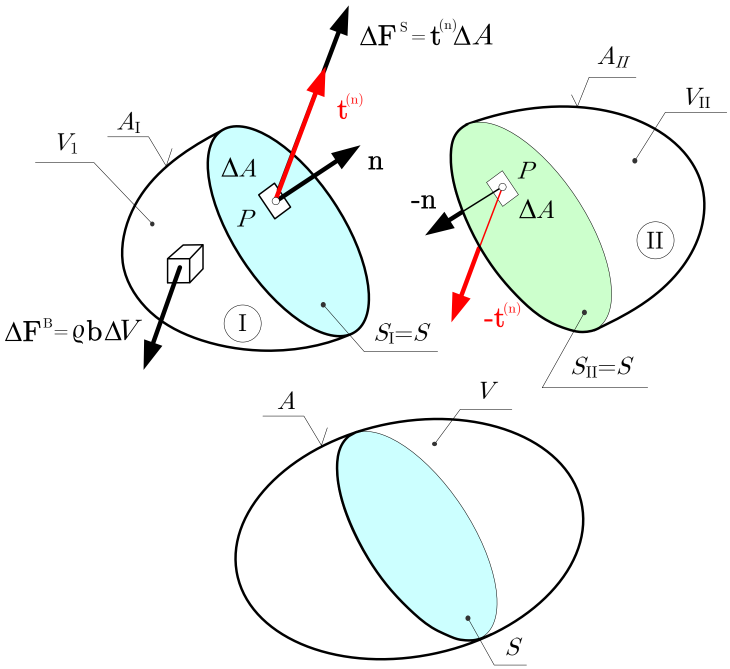

FIGURE 5.1. Inner forces acting on the surface S and body forces on the volume V

forces (or tractions) which are, in fact, contact forces. They are exerted on the body by those bodies which are in contact with the body we examine.

Assume that the resultant of the body forces acting on the volume element ΔV is $\Delta \mathbf{F}^B$. Equation

$$\rho(\mathbf{x}, t)\, \boldsymbol{b}(\mathbf{x}; t) = \lim_{\Delta V \to 0} \frac{\Delta \mathbf{F}^B}{\Delta V}\,, \tag{5.1}$$

in which $\rho(\mathbf{x}, t)$ is the density, defines the body force per

[unit volume $\rho(\mathbf{x}, t)\, \boldsymbol{b}(\mathbf{x}; t)$] {unit mass $\mathbf{b}(\mathbf{x}; t)$}.

The resultant of the external forces acting on the volume of the body is given by

$$\mathbf{F}^B = \int_V \rho\, \boldsymbol{b}\, \mathrm{d}V \tag{5.2}$$

One of the most important axioms of mechanics is the so called Cauchy's hypothesis [3], which is related to the contact forces acting on the inner surfaces of the body. The hypothesis says that at any point of time and on any inner surface element of normal $\mathbf{n}$ at the point P within the body there exists a system of surface forces.

Consider now body $\mathcal{B}$ again in the current configuration. The volume region occupied by body $\mathcal{B}$ is V with boundary $A = \partial V$. The inner surface $S = S_I = S_{II}$ slices the body into two parts denoted by V_I and V_{II}. Let $\mathbf{n}$ be the outward unit normal of surface S_I at point P. The scalar area element at P on S_I is denoted by ΔA. Let $\Delta \mathbf{F}^S$ be the resultant of the surface forces acting on the area element ΔA. The density of the surface forces at P is defined by the equation

$$\mathbf{t}(\mathbf{n}, \mathbf{x}; t) = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}^S}{\Delta A} \tag{5.3}$$

and is called stress vector. It is obvious that the stress vector inside the body depends (a) on the orientation of the surface element, i.e., on $\mathbf{n}$, (b) on the location of the point P (on $\mathbf{x}$) and (c) on time t.

The stress distribution $\mathbf{t}(\mathbf{n}, \mathbf{x}; t)$ on S_I expresses the mechanical effect of the part of the body in V_{II} on the part of the body in V_I. The mechanical effect of the part of the body in V_I on the the part of the body V_{II} is represented by the stress distribution $\mathbf{t}(-\mathbf{n}, \mathbf{x}; t)$ since the sign of the unit normal has been changed. It follows from the law of action and reaction that

$$\mathbf{t}(\mathbf{n}, \mathbf{x}; t) = -\mathbf{t}(-\mathbf{n}, \mathbf{x}; t)\,. \tag{5.4}$$

This equation shows that the stress vector is an odd function of $\mathbf{n}$.

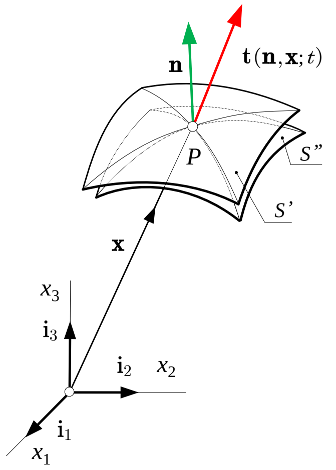

FIGURE 5.2. Stress vector on the inner surfaces S' and S''

It is worth mentioning a consequence of Cauchy's hypothesis. Namely, if we consider two or more inner surfaces which contain the point P (pass through the point P) and have the same unit normal at P – Figure 5.2 depicts two such surfaces which are denoted by S' and S'' – then the stress vector $\mathbf{t}(\mathbf{n}, \mathbf{x}; t)$ is the same at P on each inner surface.

For a fixed point of time and at a given point P of the body the stress vector depends only on the normal $\mathbf{n}$. Then we shall apply a simplified notation:

$$\mathbf{t}(\mathbf{n}, \mathbf{x}; t)|_{\mathbf{x}=const.;t=const.} = \mathbf{t}^{(n)} . \tag{5.5}$$

REMARK 5.1: According to a more general assumption to describe the mechanical interaction on the inner surfaces of the body a couple system distributed on the inner surfaces of the body might also be taken into account. Its density is denoted by $\boldsymbol{\mu}(\mathbf{n}, \mathbf{x}; t)$ and is called couple stress. It satisfies a relation similar to (5.4): $\boldsymbol{\mu}(\mathbf{n}, \mathbf{x}; t) = -\boldsymbol{\mu}(-\mathbf{n}, \mathbf{x}; t)$. This concept was introduced by the Cosserat brothers who, however, did not deal with the constitutive equations in book [35]. Making use of the concept of couple stresses various elasticity theories have been developed. In this respect it is worth citing book [55] by Witold Novaczky.

5.1.2. Cauchy's theorem. In this subsection it is our aim to determine the function $\mathbf{t}^{(n)} = \mathbf{t}^{(n)}(\mathbf{n})$. To this end consider now those conditions the small tetrahedron $PABC$ with volume ΔV removed mentally from the body $\mathcal{B}$ in the current configuration should meet in order to be in dynamic equilibrium.

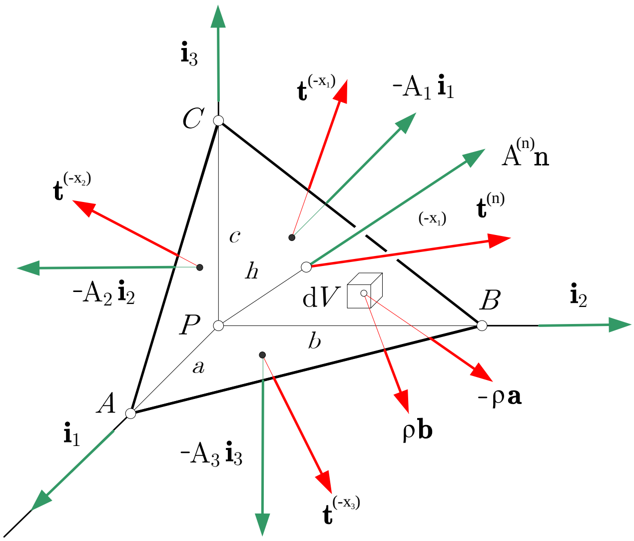

FIGURE 5.3. Tetrahedron $PABC$ with the forces acting on it

The outward unit normal to the front face ABC of the tetrahedron is denoted by $\mathbf{n}$: $\mathbf{n} = n_\ell\, \mathbf{i}_\ell$. The edges PA, PB and PC are determined by the local base vectors[1] $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$. The outward unit normals to the side faces PBC, PAC and PAB are $-\mathbf{i}_1$, $-\mathbf{i}_2$ and $-\mathbf{i}_3$. The area of the front face is denoted by $A^{(n)}$. The areas of the side faces are given by the following equations:

$$A_1 = \frac{1}{2}bc\,, \qquad A_2 = \frac{1}{2}ca\,, \qquad A_3 = \frac{1}{2}ab\,. \tag{5.6}$$

Let h be the height of the tetrahedron that belongs to the front face. With this height $\Delta V = hA^{(n)}/3$ is the volume of the tetrahedron. It is clear from Figure 5.3 that the area vector of the front face is

$$\begin{aligned} \boldsymbol{A}^{(n)} &= A^{(n)}\mathbf{n} = \frac{1}{2}\mathbf{x}_{AB} \times \mathbf{x}_{AC} = \\ &= \frac{1}{2}(-a\mathbf{i}_1 + b\mathbf{i}_2) \times (-a\mathbf{i}_1 + c\mathbf{i}_3) = \underbrace{\frac{1}{2}bc}_{A_1}\mathbf{i}_1 + \underbrace{\frac{1}{2}ca}_{A_2}\mathbf{i}_2 + \underbrace{\frac{1}{2}ab}_{A_3}\mathbf{i}_3\,. \end{aligned} \tag{5.7a}$$

Hence

$$A^{(n)}\mathbf{n} = A_\ell\, \mathbf{i}_\ell \tag{5.7b}$$

from where we get

$$\begin{gathered} A_1 = A^{(n)}\, n_1\,, \qquad A_2 = A^{(n)}\, n_2\,, \qquad A_3 = A^{(n)}\, n_3 \\ (\; A_\ell = A^{(n)}\, n_\ell \;)\,. \end{gathered} \tag{5.8}$$

[1]In Cartesian coordinate systems there is no difference between the local and global base vectors.

The densities of the surface forces acting on the side faces and the front face of the moving tetrahedron are clearly $-\mathbf{t}^{(x_1)}$, $-\mathbf{t}^{(x_2)}$, $-\mathbf{t}^{(x_3)}$ and $\mathbf{t}^{(n)}$. They are associated with the body forces $\rho\mathbf{b}$ and the effective forces $\rho\mathbf{a}$. If a body, say the tetrahedron, is in dynamic equilibrium then the external forces acting on the body are equivalent to the effective forces. A necessary condition for the equivalence is the fulfillment of equation

$$\underbrace{\int_{A^{(n)}} \mathbf{t}^{(n)}\,\mathrm{d}A - \int_{A_1} \mathbf{t}^{(x_1)}\,\mathrm{d}A - \int_{A_2} \mathbf{t}^{(x_2)}\,\mathrm{d}A - \int_{A_3} \mathbf{t}^{(x_3)}\,\mathrm{d}A + \int_V \rho\mathbf{b}\,\mathrm{d}V}_{\text{Resultant of the external forces exerted on the tetrahedron}} = \underbrace{\int_V \rho\mathbf{a}\,\mathrm{d}V}_{\substack{\text{Resultant of the}\\ \text{effective forces}}} \tag{5.9}$$

which says that the resultants of the external and effective forces are equal. Each integral in equation (5.9) can be given in a product form in which the first factor is the average value of the integrand (denoted by angle brackets), while the second one is the measure of the region over which the integral in question is taken. Consequently, we can write

$$< \mathbf{t}^{(n)} > A^{(n)} - < \mathbf{t}^{(x_1)} > A_1 - < \mathbf{t}^{(x_2)} > A_2 - < \mathbf{t}^{(x_3)} > A_3 + < \rho\mathbf{b} > \Delta V = \\ =< \rho\mathbf{a} > \Delta V\,.$$

Substituting (5.8) for A_ℓ and $hA^{(n)}/3$ for ΔV then dividing throughout by $A^{(n)}$ yield:

$$< \mathbf{t}^{(n)} > = < \mathbf{t}^{(x_1)} > n_1 + < \mathbf{t}^{(x_2)} > n_2 + < \mathbf{t}^{(x_3)} > n_3 - < \rho\mathbf{b} - \rho\mathbf{a} > h/3\,. \tag{5.10}$$

Let us now take the limit of each term in this equation by assuming that the front face of the tetrahedron moves towards P (h tends to zero) in such a manner that its orientation, i.e., $\mathbf{n}$ remains unchanged. Since both $\rho\mathbf{b}$ and $\rho\mathbf{a}$ are limited (neither the body forces nor the effective forces can be infinite) we obtain

$$\begin{gathered} \lim_{h\to 0} < \mathbf{t}^{(n)} > = \mathbf{t}^{(n)}|_P = \mathbf{t}^{(n)}\,, \\ \underbrace{\lim_{h\to 0} < \mathbf{t}^{(x_\ell)} >}_{=\mathbf{t}_\ell|_P = \mathbf{t}_\ell} n_\ell = \mathbf{t}_\ell\, n_\ell\,, \qquad \text{(no sum on } \ell) \\ \lim_{h\to 0} < \rho\mathbf{b} - \rho\mathbf{a} > h/3 = \mathbf{0}\,, \end{gathered} \tag{5.11}$$

where $\mathbf{t}_\ell$ is the stress vector at P on a plane with normal $\mathbf{i}_\ell$. Utilizing equations (5.11) we can determine the limit of equation (5.10). We have

$$\mathbf{t}^{(n)} = \mathbf{t}_\ell\, n_\ell = (\mathbf{t}_\ell \circ \mathbf{i}_\ell) \cdot \mathbf{n}\,, \tag{5.12}$$

where

$$\boxed{\boldsymbol{t} = \mathbf{t}_\ell \circ \mathbf{i}_\ell} \tag{5.13}$$

is the Cauchy stress tensor defined in the current configuration of the body. With (5.13) we can rewrite equation (5.12) into the form

$$\boxed{\mathbf{t}^{(n)} = \boldsymbol{t} \cdot \mathbf{n}\,.} \tag{5.14}$$

In words: the stress vector (Cauchy stress vector) on any surface passing through the point P is a homogeneous linear function of the normal to the surface at P. This is Cauchy's theorem.

REMARK 5.2: Cauchy published this fundamental result by investigating the equilibrium of the infinitesimal tetrahedron at the beginning of the nineteenth century [3, 5]. The condition under which Cauchy"s theorem is valid has been clarified much later by Gurtin [59, 54].

REMARK 5.3: The Cauchy stress tensor is a symmetric tensor. We shall prove this statement latter in Subsection 6.2.3 of Chapter 6.

Each stress vector ($\mathbf{t}_\ell$ in indicial notation, $\mathbf{t}_x$, $\mathbf{t}_y$ and $\mathbf{t}_z$ in the coordinate system (xyz)) can be resolved into components along the coordinate lines x_ℓ or x, y and z:

$$\mathbf{t}_\ell = t_{k\ell}\,\mathbf{i}_k\,, \qquad \mathbf{t}_\ell = \sigma_{k\ell}\,\mathbf{i}_k\,, \qquad \begin{aligned} \mathbf{t}_x &= \sigma_{xx}\mathbf{i}_x + \tau_{yx}\mathbf{i}_y + \tau_{zx}\mathbf{i}_z\,, \\ \mathbf{t}_y &= \tau_{xy}\,\mathbf{i}_x + \sigma_{yy}\mathbf{i}_y + \tau_{zy}\mathbf{i}_z\,, \\ \mathbf{t}_z &= \tau_{xz}\mathbf{i}_x + \tau_{yz}\mathbf{i}_y + \sigma_{zz}\mathbf{i}_z\,. \end{aligned} \tag{5.15}$$

Equations (5.15) reflect some notational conventions. (a) If we use indicial notations the stress components are denoted either by $t_{k\ell}$ or by $\sigma_{k\ell}$. If we use the coordinate system (xyz) – see equation $(5.15)_3$ – a further distinction is made: the stress components parallel to the normal of the surface element on which the stress vector is acting are denoted by the Greek σ and are called normal stresses: σ_{xx}, σ_{yy}, σ_{zz}. The components lying in the surface element are denoted by the Greek τ and are referred to as shear stresses: τ_{mn}, $(m, n = x, y, z;\ m \neq n)$.

In accordance with all that has just been said the matrices of the Cauchy stress tensor can be given by the following equations:

$$\underset{(3\times3)}{\underline{\mathbf{t}}} = [t_{k\ell}] = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}, \qquad \underset{(3\times3)}{\underline{\boldsymbol{\sigma}}} = [\sigma_{k\ell}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{5.16a}$$

and

$$\underset{(3\times3)}{\underline{\mathbf{t}}} = \underset{(3\times3)}{\underline{\boldsymbol{\sigma}}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}. \tag{5.16b}$$

It is worth resolving any stress vector $\mathbf{t}^{(n)}$ into two components, one parallel to the normal of the surface element and the other perpendicular to the normal $\mathbf{n}$

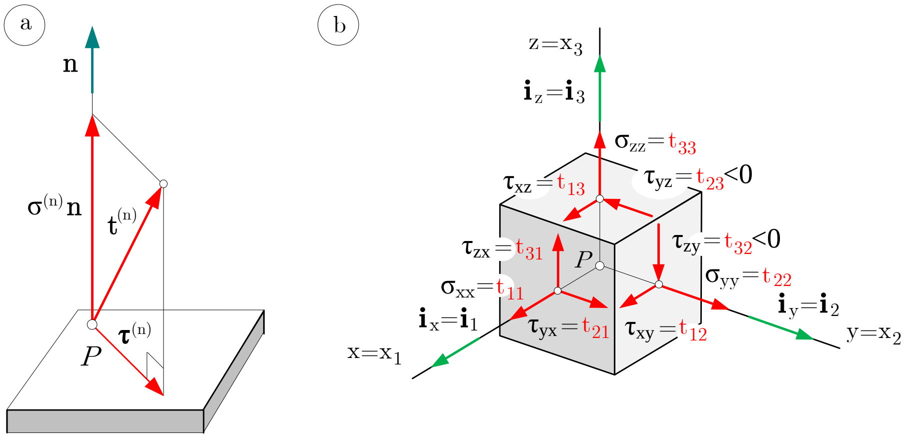

FIGURE 5.4. (a) Resolution of the stress vector into normal and shearing stresses; (b) Cubic stress element with the stresses acting on its front faces

lying, therefore, in the plane of the surface element. The normal component $\sigma^{(n)} = \sigma_{nn}$ is the normal stress, and the component $\boldsymbol{\tau}^{(n)}$ perpendicular to $\mathbf{n}$ is the shear stress (or shearing stress). Figure 5.4.a represents graphically the resolution of the stress vector $\mathbf{t}^{(n)}$. It is obvious that

$$\boxed{\sigma^{(n)} = \sigma_{nn} = \mathbf{n} \cdot \mathbf{t}^{(n)} \underset{(5.14)}{=\uparrow=} \mathbf{n} \cdot \boldsymbol{t} \cdot \mathbf{n}\,, \quad (\text{no summ on } n).} \qquad (5.17a)$$

With $\sigma^{(n)}$ we can calculate $\boldsymbol{\tau}^{(n)}$:

$$\boxed{\boldsymbol{\tau}^{(n)} = \mathbf{t}^{(n)} - \sigma^{(n)}\mathbf{n}\,, \qquad \tau^{(n)} = |\boldsymbol{\tau}^{(n)}| = \sqrt{\mathbf{t}^{(n)} \cdot \mathbf{t}^{(n)} - \left(\sigma^{(n)}\right)^2}\,.} \qquad (5.17b)$$

It is also customary to represent the stress components in the stress tensor on the front faces of a cubic stress element as shown in Figure 5.4.b. We would like to emphasize that the cube that is used to depict the stress components graphically has no dimensions: we regard it as if it were a point cube, therefore, the nine stress components shown on the cube are all acting at the same point P.

EXERCISE 5.1: Given the matrix of the Cauchy stress tensor in the coordinate system $(x_1x_2x_3)$. What is the dyadic form of the tensor? Show the stress components on a stress element. Determine the stress vector and the normal and shearing stresses on a plane with normal $\mathbf{n}$.

$$\underset{(3\times3)}{\underline{\mathbf{t}}} = \left[\ \underset{(3\times1)}{\underline{\mathbf{t}_1}}\ \middle|\ \underset{(3\times1)}{\underline{\mathbf{t}_2}}\ \middle|\ \underset{(3\times1)}{\underline{\mathbf{t}_3}}\ \right] = \begin{bmatrix} 92 & -20 & 0 \\ -20 & -4 & 0 \\ 0 & 0 & -40 \end{bmatrix} \left[\mathrm{N/mm^2}\right],$$

$$\mathbf{n} = \frac{1}{\sqrt{26}}(5\mathbf{i}_1 - \mathbf{i}_2)\,.$$

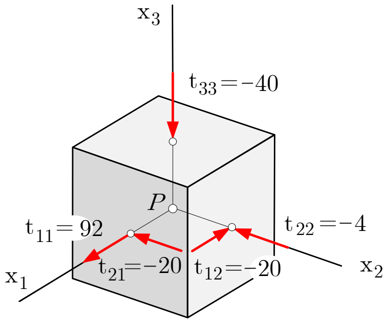

FIGURE 5.5. The cubic stress element for the given stress tensor

Making use of equation (5.13) we can write

$$\boldsymbol{t} = \mathbf{t}_\ell \circ \mathbf{i}_\ell = (92\mathbf{i}_1 - 20\mathbf{i}_2) \circ \mathbf{i}_1 - (20\mathbf{i}_1 + 4\mathbf{i}_2) \circ \mathbf{i}_2 - 40\mathbf{i}_3 \circ \mathbf{i}_3 \quad \left[\frac{\mathrm{N}}{\mathrm{mm}^2}\right] . \tag{5.18}$$

The non zero stress components are represented on the stress element. Equations (5.14) and (5.17a) yield the stress vector and the normal stress:

$$\underline{\mathbf{t}}^{(n)} = \underline{\mathbf{t}}\,\underline{\mathbf{n}} = \frac{1}{\sqrt{26}} \begin{bmatrix} 92 & -20 & 0 \\ -20 & -4 & 0 \\ 0 & 0 & -40 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{26}} \begin{bmatrix} 480 \\ -96 \\ 0 \end{bmatrix} \left[\mathrm{N/mm}^2\right],$$
$$\sigma^{(n)} = \mathbf{t}^{(n)} \cdot \mathbf{n} = \frac{1}{26}(5 \times 480 + 96) = 96 \left[\mathrm{N/mm}^2\right].$$

With $\underline{\mathbf{t}}^{(n)}$ and $\sigma^{(n)}$ we can calculate the stress vector by utilizing equation (5.17b):

$$\underline{\boldsymbol{\tau}}^{(n)} = \underline{\mathbf{t}}^{(n)} - \sigma^{(n)}\,\underline{\mathbf{n}} = \frac{1}{\sqrt{26}} \begin{bmatrix} 480 \\ -96 \\ 0 \end{bmatrix} - \frac{96}{\sqrt{26}} \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\boldsymbol{\tau}^{(n)} = \mathbf{0}$ it follows that the normal $\mathbf{n}$ and the stress vector $\mathbf{t}^{(n)} = \sigma^{(n)}\mathbf{n}$ are parallel to each other, therefore the direction $\mathbf{n}$ is a principal direction.

EXERCISE 5.2: What is the matrix of the Cauchy stress tensor in a cylindrical coordinate system?

Recalling equations (1.190), (1.191) and (1.192) we have

$$\boldsymbol{t} = \mathbf{t}_R \circ \mathbf{e}_R + \mathbf{t}_\vartheta \circ \mathbf{e}_\vartheta + \mathbf{t}_z \circ \mathbf{e}_z \tag{5.19a}$$

in which

$$\begin{aligned} \mathbf{t}_R &= t_{RR}\,\mathbf{e}_R + t_{\vartheta R}\,\mathbf{e}_\vartheta + t_{zR}\,\mathbf{e}_z = \sigma_{RR}\,\mathbf{e}_R + \tau_{\vartheta R}\,\mathbf{e}_\vartheta + \tau_{zR}\,\mathbf{e}_z\,, \\ \mathbf{t}_\vartheta &= t_{R\vartheta}\,\mathbf{e}_R + t_{\vartheta\vartheta}\,\mathbf{e}_\vartheta + t_{z\vartheta}\,\mathbf{e}_z = \tau_{R\vartheta}\,\mathbf{e}_R + \sigma_{\vartheta\vartheta}\,\mathbf{e}_\vartheta + \tau_{z\vartheta}\,\mathbf{e}_z\,, \\ \mathbf{t}_z &= t_{Rz}\,\mathbf{e}_R + t_{\vartheta z}\,\mathbf{e}_\vartheta + t_{zz}\,\mathbf{e}_z = \tau_{Rz}\,\mathbf{e}_R + \tau_{\vartheta z}\,\mathbf{e}_\vartheta + \sigma_{z\vartheta}\,\mathbf{e}_z \end{aligned} \tag{5.19b}$$

are the three stress vectors. Hence

$$\underset{(R\vartheta z)}{\mathbf{t}} = \left[\underset{(R\vartheta z)}{\mathbf{t}_R} \;\middle|\; \underset{(R\vartheta z)}{\mathbf{t}_\vartheta} \;\middle|\; \underset{(R\vartheta z)}{\mathbf{t}_z} \right] = \begin{bmatrix} t_{RR} & t_{R\vartheta} & t_{Rz} \\ t_{\vartheta R} & t_{\vartheta\vartheta} & t_{\vartheta z} \\ t_{zR} & t_{z\vartheta} & t_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{RR} & \tau_{R\vartheta} & \tau_{Rz} \\ \tau_{\vartheta R} & \sigma_{\vartheta\vartheta} & \tau_{\vartheta z} \\ \tau_{zR} & \tau_{z\vartheta} & \sigma_{zz} \end{bmatrix} \tag{5.20}$$

is the matrix sought.

5.2. Further stress tensors

5.2.1. The Kirchhoff stress tensor. The Kirchoff stress tensor is defined by the following equation:

$$\boxed{\boldsymbol{\tau} = J\boldsymbol{t}\,, \qquad \tau_{k\ell} = Jt_{k\ell}\,.} \tag{5.21}$$

5.2.2. The first Piola-Kirchhoff stress tensor. Figure 5.6 shows an infinitesimal surface element before deformation (in the initial configuration) and at time t (in the current configuration).

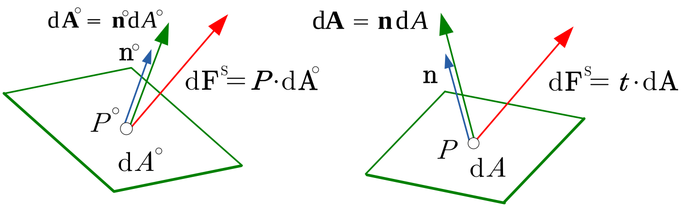

FIGURE 5.6. Surface element before deformation and at time t

The resultant of the Cauchy stresses acting on the surface element $\mathrm{d}A$ is given by the relation

$$\mathrm{d}\mathbf{F}^S = \boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A = \boldsymbol{t}\cdot\mathrm{d}\mathbf{A}\,. \tag{5.22}$$

Recalling Nanson's formula – see equation (2.89) – we can rewrite the above equation

$$\begin{aligned} \mathrm{d}\mathbf{F}^S &= J\,\boldsymbol{t}\cdot\boldsymbol{F}^{-T}\cdot\mathrm{d}\mathbf{A}^\circ = J\,\boldsymbol{t}\cdot\boldsymbol{F}^{-T}\cdot\mathbf{n}^\circ\mathrm{d}A^\circ\,, \\ \mathrm{d}F^S_k &= J\,t_{k\ell}F^{-1}_{\ell B}n^\circ_B\mathrm{d}A^\circ\,. \end{aligned} \tag{5.23}$$

On the basis of this result we define the first Piola-Kirchhoff[2] stress tensor [12, 22], [87], [6, 8, 11] by the following relation

$$\boxed{\boldsymbol{P} = J\,\boldsymbol{t}\cdot\boldsymbol{F}^{-T}, \qquad P_{kB} = J\,t_{k\ell}F^{-1}_{\ell B}\,.} \tag{5.24}$$

REMARK 5.4: The first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ is a two point tensor. If the tensor $\boldsymbol{P}$ is known we can calculate the force $\mathrm{d}\mathbf{F}^S$ by using those geometrical data which belong to the initial configuration. It is also worthy of mentioning that the first Piola-Kirchhoff stress tensor is not symmetric tensor.

[2] Gustav Robert Kirchhoff, 1824-1887

5.2.3. The second Piola-Kirchhoff stress tensor. Let $\mathrm{d}\mathbf{F}^{\circ S}$ be a fictitious force acting on the surface element $\mathrm{d}A^\circ$ which belongs to the initial configuration. We shall assume that

$$\mathrm{d}\mathbf{F}^{\circ S} \ \text{ is related to } \ \mathrm{d}\mathbf{F}^{S}$$

in the same manner as

$$\mathrm{d}\mathbf{X} \ \text{ is related to } \ \mathrm{d}\mathbf{x}$$

$$\left(\ \mathrm{d}\mathbf{X} = \boldsymbol{F}^{-1} \cdot \mathrm{d}\mathbf{x} \ \right) ,$$

that is,

$$\mathrm{d}\mathbf{F}^{\circ S} = \boldsymbol{F}^{-1} \cdot \mathrm{d}\mathbf{F}^{S} . \tag{5.25}$$

If we now substitute $(5.23)_1$ for $\mathrm{d}\mathbf{F}^S$ in the above equation we have

$$\mathrm{d}\mathbf{F}^{\circ S} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot \boldsymbol{F}^{-T} \cdot \mathbf{n}^\circ \mathrm{d}A^\circ \tag{5.26}$$

in which

$$\boxed{\boldsymbol{S} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot \boldsymbol{F}^{-T} = \boldsymbol{F}^{-1} \cdot \boldsymbol{P} , \quad S_{AB} = JF^{-1}_{Ak}\, t_{k\ell}\, F^{-1}_{\ell B} = F^{-1}_{Ak} P_{kB}} \tag{5.27}$$

is the second Piola-Kirchhoff stress tensor.

Remark 5.5: The stress vector

$$\mathbf{t}^{\circ(n)} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot \boldsymbol{F}^{-T} \cdot \mathbf{n}^\circ = \boldsymbol{S} \cdot \mathbf{n}^\circ , \quad t^{\circ(n)}_A = JF^{-1}_{Ak}\, t_{k\ell}\, F^{-1}_{\ell B} n^\circ_B = S_{AB} n^\circ_B \tag{5.28}$$

is a pseudo stress vector. Since $\mathrm{d}\mathbf{F}^{\circ S} = \mathbf{t}^{\circ(n)}\,\mathrm{d}A^\circ$ and $\mathrm{d}\mathbf{F}^S = \mathbf{t}^{(n)}\,\mathrm{d}A$ it follows from equation (5.25) that

$$\mathbf{t}^{\circ(n)} = \boldsymbol{F}^{-1} \cdot \mathbf{t}^{(n)} \frac{\mathrm{d}A}{\mathrm{d}A^\circ} \quad \text{and} \quad \mathbf{t}^{(n)} = \boldsymbol{F} \cdot \mathbf{t}^{\circ(n)} \frac{\mathrm{d}A^\circ}{\mathrm{d}A} = \boldsymbol{F} \cdot \boldsymbol{S} \cdot \mathbf{n}^\circ \frac{\mathrm{d}A^\circ}{\mathrm{d}A} . \tag{5.29}$$

Remark 5.6: The second Piola-Kirchhoff stress tensor belongs to the initial configuration. It follows from its definition that $\boldsymbol{S} = \boldsymbol{S}^T$, i.e., the tensor is symmetric.

Remark 5.7: In the linear theory of deformations we can simplify the relations that define the first and second Piola-Kirchhoff stress tensors. Recalling equations (5.24) and (5.27) we may write

$$\boldsymbol{P} = J\,\boldsymbol{t} \cdot \boldsymbol{F}^{-T} = \underset{(4.28)}{\uparrow} \ \underset{(4.3)}{\uparrow} \approx (1 + \varepsilon_I)\ \boldsymbol{t} \cdot (\boldsymbol{1} - \nabla \circ \mathbf{u}) \approx \boldsymbol{t} = \boldsymbol{\sigma} \tag{5.30}$$

and

$$\boldsymbol{S} = J\,\boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot \boldsymbol{F}^{-T} \approx (1 + \varepsilon_I)\ (\boldsymbol{1} - \mathbf{u} \circ \nabla) \cdot \boldsymbol{t} \cdot (\boldsymbol{1} - \nabla \circ \mathbf{u}) \approx \boldsymbol{t} = \boldsymbol{\sigma} . \tag{5.31}$$

Hence, we can assume that the three stress tensors are the same within the framework of the linear deformation theory. In accordance with (5.30) and (5.31) we shall denote it by $\boldsymbol{\sigma}$.

5.2.4. The Biot stress tensor. The Biot stress tensor [50] is defined by the equation:

$$\boxed{\boldsymbol{T} = \frac{1}{2}\left(\boldsymbol{U}\cdot\boldsymbol{S} + \boldsymbol{S}\cdot\boldsymbol{U}\right), \qquad T_{AB} = \frac{1}{2}\left(U_{AL}S_{LB} + S_{AL}U_{LB}\right).} \tag{5.32}$$

The Biot stress tensor is a symmetric tensor in the initial configuration.

5.2.5. The Kirchhoff stress tensor. The Kirchhoff stress tensor is directly proportional to the Cauchy stress tensor:

$$\boxed{\boldsymbol{\tau} = J\boldsymbol{t}, \qquad \tau_{k\ell} = Jt_{k\ell}.} \tag{5.33}$$

It is obvious that the Kirchhoff stress tensor is symmetric.

5.2.6. Nominal stresses. The nominal stress tensor is defined by the following relationship:

$$\boxed{\boldsymbol{\mathcal{N}} = \boldsymbol{F}^{-1}\cdot\boldsymbol{t}, \qquad \mathcal{N}_{A\ell} = F^{-1}_{Ak}t_{k\ell}.} \tag{5.34}$$

The nominal stress tensor is not symmetric.

5.3. Extreme values of the normal and shearing stresses

5.3.1. Extreme values of the normal stresses, principal directions. For a fixed point of time t the Cauchy stress $\mathbf{t}^{(n)}$, the normal stress $\sigma^{(n)}$ and the shearing stress $\boldsymbol{\tau}^{(n)}$ are functions of the surface normal at the point P of the current configuration. In the case their extreme values are known we shall be able to decide whether the material of the body is capable of resisting to the stress state at P without any damage.

We seek, therefore, the extreme values of the function $\sigma^{(n)} = \sigma^{(n)}(\mathbf{n}) = \mathbf{n}\cdot\boldsymbol{t}\cdot\mathbf{n}$ under the side condition $|\mathbf{n}| = 1$. (The normal vectors $\mathbf{n}$ are centered at P and their tips are located on a sphere of unit radius.) The side condition can be taken into account by the Lagrange multiplier method. Consequently, it is our aim to find the stationary conditions for the Lagrange function

$$\mathcal{L}(\mathbf{n},\sigma) = \mathbf{n}\cdot\boldsymbol{t}\cdot\mathbf{n} - \sigma\left(\mathbf{n}^2 - 1\right), \tag{5.35}$$

where the Lagrange multiplier is denoted simply by σ. Changes of $\mathbf{n}$ are characterized by the operator

$$\nabla^{(n)} = \frac{\partial}{\partial n_1}\mathbf{i}_1 + \frac{\partial}{\partial n_2}\mathbf{i}_2 + \frac{\partial}{\partial n_3}\mathbf{i}_3 . \tag{5.36}$$

It is not difficult to check that

$$\mathbf{n}\circ\nabla^{(n)} = (n_1\,\mathbf{i}_1 + n_2\,\mathbf{i}_2 + n_3\,\mathbf{i}_3)\circ\nabla^{(n)} = \boldsymbol{1} . \tag{5.37}$$

By taking the above relation into account we have

$$\mathcal{L}(\mathbf{n},\sigma)\,\nabla^{(n)} = \overset{\downarrow}{\mathbf{n}}\cdot\boldsymbol{t}\cdot\mathbf{n}\,\nabla^{(n)} + \mathbf{n}\cdot\boldsymbol{t}\cdot\overset{\downarrow}{\mathbf{n}}\,\nabla^{(n)} - 2\sigma\mathbf{n}\cdot\left(\mathbf{n}\circ\nabla^{(n)}\right) =$$
$$= 2\mathbf{n}\cdot\underbrace{\boldsymbol{t}\cdot\left(\mathbf{n}\circ\nabla^{(n)}\right)}_{\boldsymbol{t}} - 2\sigma\mathbf{n}\cdot\boldsymbol{1} = 2\left(\boldsymbol{t} - \sigma\boldsymbol{1}\right)\cdot\mathbf{n} = \mathbf{0}$$

or

$$\boxed{(\boldsymbol{t} - \sigma \boldsymbol{1}) \cdot \mathbf{n} = \mathbf{0}} \tag{5.38}$$

which is the first stationary condition. Note that the down-arrow shows the quantity the operator $\nabla^{(n)}$ is applied to. It should also be mentioned that we have utilized the symmetry of the Cauchy stress tensor in the transformations.

The second stationary condition is obtained if we derive the Lagrange function (5.35) with respect to the multiplier σ. The result is the side condition for $\mathbf{n}$:

$$\frac{\partial \mathcal{L}}{\partial \sigma} = \mathbf{n}^2 - 1 = 0\,. \tag{5.39}$$

EXERCISE 5.3: Derive the stationary conditions for the normal stress $\sigma^{(n)}$ in indicial notation.

It is easy to check with regard to (5.35) that

$$\mathcal{L}(\mathbf{n}, \sigma) = n_k t_{k\ell} n_\ell - \sigma\,(n_\ell n_\ell - 1) \tag{5.40}$$

is the Lagrange function. If we take the symmetry condition $t_{k\ell} = t_{\ell k}$ into account and write k for ℓ and ℓ for k where necessary, the first stationary condition yields

$$\begin{aligned} \frac{\partial \mathcal{L}(n_s, \sigma)}{\partial n_r} &= \underbrace{\frac{\partial n_k}{\partial n_r}}_{\delta_{kr}} t_{k\ell} n_\ell + n_k t_{k\ell} \underbrace{\frac{\partial n_\ell}{\partial n_r}}_{\delta_{r\ell}} - 2\sigma \underbrace{\frac{\partial n_\ell}{\partial n_r}}_{\delta_{r\ell}} = \\ &= \delta_{kr} t_{k\ell} n_\ell + \underbrace{n_k t_{k\ell} \delta_{\ell r}}_{= n_\ell t_{\ell k} \delta_{kr} = \delta_{kr} t_{k\ell} n_\ell} - 2\delta_{r\ell} n_\ell = \\ &= 2 t_{r\ell} n_\ell - 2\sigma\, \delta_{r\ell} n_\ell = 2\,(t_{r\ell} - \sigma\, \delta_{r\ell})\, n_\ell = 0 \end{aligned}$$

or

$$(t_{r\ell} - \sigma\, \delta_{r\ell})\, n_\ell = 0\,. \tag{5.41}$$

The second stationary condition is again the side condition:

$$\frac{\partial \mathcal{L}(n_s, \sigma)}{\partial n_r} = n_\ell n_\ell - 1 = 0\,. \tag{5.42}$$

According to the stationary conditions (5.38) (or the equivalent stationary condition (5.41)) the principal values of the eigenvalue problem of the symmetric Cauchy stress tensor give the extreme values of the normal stresses. The elements in the ordered set $\sigma_1 \geq \sigma_2 \geq \sigma_3$ of the eigenvalues are called principal stresses, the corresponding principal directions are named principal stress directions. These are given by the eigenvectors $\mathbf{n}_\ell$, $\ell = 1, 2, 3$.

Let us denote the scalar invariants of the Cauchy stress tensor by T_I, T_{II} and T_{III}. It follows from (1.113) that

$$T_I = t_{\ell\ell} = t_{11} + t_{22} + t_{33}\,, \tag{5.43a}$$

$$T_{II} = \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix} + \begin{vmatrix} t_{11} & t_{13} \\ t_{31} & t_{33} \end{vmatrix} + \begin{vmatrix} t_{22} & t_{23} \\ t_{32} & t_{33} \end{vmatrix} = \frac{1}{2}\,(t_{kk} t_{\ell\ell} - t_{k\ell} t_{\ell k})\,, \tag{5.43b}$$

$$T_{III} = \det(t_{k\ell}) = \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix} = e_{k\ell r} t_{1k} t_{2\ell} t_{3r} \tag{5.43c}$$

are the scalar invariants. After having the scalar invariants determined we obtain, by utilizing (1.112b), the characteristic equation

$$P_3(\sigma) = -\det(\underline{\mathbf{t}} - \sigma \underline{\mathbf{1}}) = -\begin{vmatrix} t_{11}-\sigma & t_{12} & t_{13} \\ t_{21} & t_{22}-\sigma & t_{23} \\ t_{31} & t_{32} & t_{33}-\sigma \end{vmatrix} =$$
$$= \sigma^3 - T_I\sigma^2 + T_{II}\sigma - T_{III} = 0\,. \tag{5.44}$$

The roots of the characteristic equation are the principal stresses, i.e., the extreme values of the normal stress $\sigma^{(n)}$. Having determined the roots we can calculate the principal stress directions by solving equation system (5.38) (or (5.41)) for $\mathbf{n}$ (n_ℓ).

Recalling (1.116) and (1.118) we get the spectral decomposition and the matrix of the Cauchy stress tensor in the Cartesian coordinate system constituted by the principal stress directions:

$$\underset{(n_1 n_2 n_3)}{\boldsymbol{t}} = \sum_{\ell=1}^{3} \sigma_\ell \, \mathbf{n}_\ell \otimes \mathbf{n}_\ell\,, \qquad \underset{(n_1 n_2 n_3)}{\underline{\mathbf{t}}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}. \tag{5.45}$$

Exercise 5.4: Given the Cauchy stress tensor in a dyadic form:

$$\boldsymbol{t} = \alpha x_2\, \mathbf{i}_3 \circ \mathbf{i}_1 - \beta x_1\, \mathbf{i}_3 \circ \mathbf{i}_2 + (\alpha x_2\, \mathbf{i}_1 - \beta x_1\, \mathbf{i}_2) \circ \mathbf{i}_3\,,$$

where α and β are positive constants. Determine (a) the scalar invariants, (b) the principal stresses and (c) the principal stress directions at the point $(\alpha, \beta, 0)$.

We can read off from the characteristic equation

$$P_3(\sigma) = -\begin{vmatrix} -\sigma & 0 & \alpha\beta \\ 0 & -\sigma & -\alpha\beta \\ \alpha\beta & -\alpha\beta & -\sigma \end{vmatrix} = \sigma^3 - T_I\sigma^2 + T_{II}\sigma - T_{III} = \sigma^3 - 2\alpha^2\beta^2\sigma = 0$$

that

$$T_I = 0\,, \qquad T_{II} = -2\alpha^2\beta^2\,, \qquad T_{III} = 0\,.$$

Hence

$$\sigma_1 = \sqrt{2}\beta\alpha, \qquad \sigma_2 = 0, \qquad \sigma_3 = -\sqrt{2}\beta\alpha$$

are the three principal stresses. If $\ell = 2$ the equation system

$$(\boldsymbol{t} - \sigma_\ell \boldsymbol{1}) \cdot \mathbf{n} = \mathbf{0}\,, \qquad \begin{bmatrix} -\sigma_\ell & 0 & \alpha\beta \\ 0 & -\sigma_\ell & -\alpha\beta \\ \alpha\beta & -\alpha\beta & -\sigma_\ell \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \ell = 1, 2, 3$$

which gives the principal stress directions, yields by taking the relation $\sigma_2 = 0$ into account that

$$\mathbf{n}_2 = \frac{1}{\sqrt{2}} (\mathbf{i}_1 + \mathbf{i}_2)\,, \qquad |\mathbf{n}_2| = 1\,.$$

If $\ell = 1$ $(\sigma_1 = \sqrt{2}\beta\alpha)$ we obtain the following homogeneous equation system for calculating the first principal stress direction:

$$\sqrt{2}n_1 + n_3 = 0\,, \quad \sqrt{2}n_2 - n_3 = 0\,, \quad n_1 - n_2 + \sqrt{2}n_3 = 0\,.$$

The solution is of the form

$$\mathbf{n}_1 = \frac{1}{2}\left(\mathbf{i}_1 - \mathbf{i}_2 + \sqrt{2}\mathbf{i}_3\right)\,, \qquad |\mathbf{n}_2| = 1\,.$$

With the eigenvectors $\mathbf{n}_1$ and $\mathbf{n}_2$

$$\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2 = \frac{1}{2}\left(-\mathbf{i}_1 + \mathbf{i}_2 + \sqrt{2}\mathbf{i}_3\right)$$

is the third eigenvector.

5.3.2. Extreme values of the shearing stresses. We shall determine the extreme values of the shearing stresses in the Cartesian coordinate system $(\xi\eta\zeta)$ constituted by the principal axes. In this coordinate system the coordinate axes ξ, η és ζ are the first, second and third principal directions:

$$\mathbf{i}_\xi = \mathbf{n}_1\,, \quad \mathbf{i}_\eta = \mathbf{n}_2\,, \quad \mathbf{i}_\zeta = \mathbf{n}_3\,. \tag{5.46}$$

Hence

$$\boldsymbol{t} = \sigma_1\,\mathbf{i}_\xi \circ \mathbf{i}_\xi + \sigma_2\,\mathbf{i}_\eta \circ \mathbf{i}_\eta + \sigma_3\,\mathbf{i}_\zeta \circ \mathbf{i}_\zeta \quad \text{and} \quad \underset{(\xi\eta\zeta)}{\underline{\mathbf{t}}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{5.47}$$

are the Cauchy stress tensor and its matrix. The unit normal to the surface on which the shearing stress has an extreme value is of the form

$$\mathbf{n} = n_\xi\,\mathbf{i}_\xi + n_\eta\,\mathbf{i}_\eta + n_\zeta\mathbf{i}_\zeta\,, \qquad |\mathbf{n}| = 1 \tag{5.48}$$

in the coordinate system $(\xi\eta\zeta)$. Consequently,

$$\begin{aligned} \mathbf{t}^{(n)} &= \boldsymbol{t}\cdot\mathbf{n} = \sigma_1 n_\xi\,\mathbf{i}_\xi + \sigma_2 n_\eta\,\mathbf{i}_\eta + \sigma_3 n_\zeta\,\mathbf{i}_\zeta\,, \\ \sigma^{(n)} &= \mathbf{n}\cdot\boldsymbol{t}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{t}_n = \sigma_1 n_\xi^2 + \sigma_2 n_\eta^2 + \sigma_3 n_\zeta^2 \end{aligned} \tag{5.49}$$

are the Cauchy stress vector and the normal stress. Making use of equation (5.17b) we can determine the square of the shearing stress:

$$\begin{aligned} \left(\tau^{(n)}\right)^2 &= (\mathbf{t}^{(n)} - \sigma^{(n)}\mathbf{n})\cdot(\mathbf{t}^{(n)} - \sigma^{(n)}\mathbf{n}) = \\ &= |\mathbf{t}^{(n)}|^2 - (\sigma^{(n)})^2 - \underbrace{(\mathbf{t}^{(n)} - \sigma^{(n)}\mathbf{n})\cdot\sigma^{(n)}\mathbf{n}}_{=0} = |\mathbf{t}^{(n)}|^2 - (\sigma^{(n)})^2 = \\ &= (\sigma_1 n_\xi)^2 + (\sigma_2 n_\eta)^2 + (\sigma_3 n_\zeta)^2 - \left(\sigma_1 n_\xi^2 + \sigma_2 n_\eta^2 + \sigma_3 n_\zeta^2\right)^2\,. \end{aligned} \tag{5.50}$$

It is our aim to find the extreme values of this quantity under the side condition

$$n_\xi^2 + n_\eta^2 + n_\zeta^2 = 1\,. \tag{5.51}$$

Let us substitute n_ζ^2 from here into (5.50). We obtain

$$\begin{aligned}(\tau^{(n)})^2(n_\xi, n_\eta) &= \\ &= \left[(\sigma_1)^2 - (\sigma_3)^2\right] n_\xi^2 + \left[(\sigma_2)^2 - (\sigma_3)^2\right] n_\eta^2 - \left[(\sigma_1 - \sigma_3) n_\xi^2 + (\sigma_2 - \sigma_3) n_\eta^2\right]^2 .\end{aligned} \tag{5.52}$$

Thus

$$\begin{aligned}\frac{\partial \tau_n^2}{\partial n_\xi} &= 2n_\xi(\sigma_1 - \sigma_3)\left\{\sigma_1 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3)\, n_\xi^2 + (\sigma_2 - \sigma_3)\, n_\eta^2\right]\right\} = 0\,, \\ \frac{\partial \tau_n^2}{\partial n_\eta} &= 2n_\eta(\sigma_2 - \sigma_3)\left\{\sigma_2 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3)\, n_\xi^2 + (\sigma_2 - \sigma_3)\, n_\eta^2\right]\right\} = 0\end{aligned} \tag{5.53}$$

are the necessary conditions for an extremum. A possible solution is $n_\xi = n_\eta = 0$, $n_\zeta = \pm 1$. Let us repeat the previous steps by substituting first n_η, second n_ξ from (5.51) into the formula for $\left(\tau^{(n)}\right)^2$. We obtain the solutions $n_\xi = n_\zeta = 0$, $n_\eta = \pm 1$ and $n_\eta = n_\zeta = 0$, $n_\xi = \pm 1$. The three solutions we have determined are the principal directions of the stress tensor which shows that the three shearing stresses are all equal to zero: we have found local minimums.

Further solutions can be obtained from (5.53) if we assume merely that $n_\xi = 0$. Then equation $(5.53)_2$ yields $n_\eta = \pm 1/\sqrt{2}$. Making use of these solutions from side condition (5.51) we get the third component of $\mathbf{n}$: $n_\zeta = \pm 1/\sqrt{2}$. After setting n_η to zero equation $(5.53)_1$ and side condition (5.51) result in the values $n_\xi = \pm 1/\sqrt{2}$ and $n_\zeta = \pm 1/\sqrt{2}$. For the case $n_\zeta = 0$ the first in the equation pair $\partial \tau_n^2 / \partial n_\eta = 0$ and $\partial \tau_n^2 / \partial n_\zeta = 0$ together with side condition (5.51) lead to the result $n_\eta = \pm 1/\sqrt{2}$ and $n_\xi = \pm 1/\sqrt{2}$ – the line of thought is the same as before.

A summary of the results is given by the table below: the equation rows contain the normal sought and the square of the shearing stress that belongs to it:

$$\boxed{\begin{aligned}\mathbf{n} &= \pm\frac{1}{\sqrt{2}}(\mathbf{i}_\eta - \mathbf{i}_\zeta)\,, \quad (\tau^{(n)})^2 = \frac{1}{4}(\sigma_2 - \sigma_3)^2\,, \\ \mathbf{n} &= \pm\frac{1}{\sqrt{2}}(\mathbf{i}_\xi - \mathbf{i}_\zeta)\,, \quad (\tau^{(n)})^2 = \frac{1}{4}(\sigma_1 - \sigma_3)^2\,, \\ \mathbf{n} &= \pm\frac{1}{\sqrt{2}}(\mathbf{i}_\xi - \mathbf{i}_\eta)\,, \quad (\tau^{(n)})^2 = \frac{1}{4}(\sigma_1 - \sigma_2)^2\,.\end{aligned}} \tag{5.54}$$

We remark that $(\tau^{(n)})^2$ is calculated by using equation (5.50). If we take into account that the principal stresses form an ordered set $(\sigma_1 \geq \sigma_2 \geq \sigma_3)$ we can read off from the above relations that

$$\tau_{\max}^{(n)} = \frac{1}{2}|\sigma_1 - \sigma_3| \tag{5.55a}$$

is the maximum shearing stress. Assuming a positive sign

$$\mathbf{n} = \pm\frac{1}{\sqrt{2}}(\mathbf{i}_\xi - \mathbf{i}_\zeta)\,, \tag{5.55b}$$

is the normal to the surface element on which $\tau_{\max}^{(n)}$ is the maximum shearing stress. With this normal, equations (5.49) yield the Cauchy stress vector and the normal stress on this surface element:

$$\begin{aligned} \mathbf{t}^{(n)} &= \boldsymbol{t}\cdot\mathbf{n} = \sigma_1 n_\xi\,\mathbf{i}_\xi + \sigma_2 n_\eta\,\mathbf{i}_\eta + \sigma_3 n_\zeta\,\mathbf{i}_\zeta = \frac{1}{\sqrt{2}}\left(\sigma_1\,\mathbf{i}_\xi - \sigma_3\,\mathbf{i}_\zeta\right)\,, \\ \sigma^{(n)} &= \mathbf{n}\cdot\boldsymbol{t}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{t}_n = \sigma_1 n_\xi^2 + \sigma_2 n_\eta^2 + \sigma_3 n_\zeta^2 = \frac{1}{2}\left(\sigma_1 + \sigma_3\right)\,. \end{aligned} \tag{5.56}$$

5.4. Boundary and initial conditions

The *boundary conditions* prescribe the values of some physical quantities (vector and/or tensor fields) on the boundary of the body.

The *initial conditions* are prescriptions on the displacement and velocity fields at the beginning of the motion (in the initial configuration).

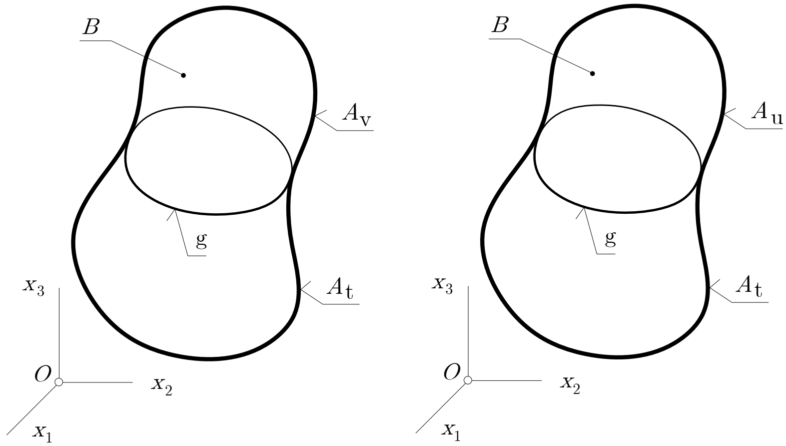

FIGURE 5.7. The parts of the boundary surface A

Assume that the boundary surface A of the body is divided into two parts $A_u = A_v$ and A_t which are separated from each other by the curve g: $A_u \cup A_t = A$; $A_u \cap A_t = 0$. In a limit case $A_u = A_v = A$ (or $A_t = A$).

Prescribed quantities on the boundary will be denoted by a tilde over the letter that identifies the quantity in question.

We speak about kinematic boundary conditions if the (velocities) [displacements] are prescribed on (A_v) $[A_u]$. The kinematic (or displacement) boundary conditions have the form

$$\mathbf{v} = \tilde{\mathbf{v}}\,, \qquad v_\ell = \tilde{v}_\ell\,, \qquad \forall \mathbf{x} \in A_v \tag{5.57}$$

$$\mathbf{u} = \tilde{\mathbf{u}}\,, \qquad u_\ell = \tilde{u}_\ell\,. \qquad \forall \mathbf{x} \in A_u \tag{5.58}$$

We speak about dynamic (or traction) boundary conditions if the stress vector (the traction vector) is prescribed on A_t.

$$\mathbf{t}^{(n)} = \boldsymbol{t}\cdot\mathbf{n} = \tilde{\mathbf{t}}\,, \qquad t_k^{(n)} = t_{k\ell} n_\ell = \tilde{t}_k\,. \qquad \forall \mathbf{x} \in A_t \tag{5.59}$$

The boundary value problem is of (Dirichlet) [Neumann] type if (displacements) [tractions] are prescribed on the whole boundary.

A further type of boundary conditions is constituted by the mixed boundary conditions. Then we make partly kinematic, partly dynamic prescriptions on the very same part of the boundary. Consider a part of the boundary $A_{m1} \in A$ and assume that tractions parallel to the normal (denoted by $\tilde{t}_{||}$) and displacements perpendicular to the normal (lying therefore in the tangent plane and denoted by $\tilde{\mathbf{u}}_{\perp}$) are prescribed on A_{m1}. If this is the case the boundary conditions take the forms

$$\mathbf{n} \cdot \boldsymbol{t} \cdot \mathbf{n} = \tilde{t}_{||}\,, \qquad \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) = \mathbf{u} - \mathbf{n}\,(\mathbf{u} \cdot \mathbf{n}) = \tilde{\mathbf{u}}_{\perp}\,, \quad \forall \mathbf{x} \in A_{m1}\,. \tag{5.60}$$

Consider now the part of the boundary $A_{m2} \in A$ and assume that the displacement in the normal direction (denoted by $\tilde{u}_{||}$) and stresses in the tangent plane (denoted by $\tilde{\mathbf{t}}_{\perp}$ - these are shearing stresses) are prescribed on A_{m2}. Then

$$\mathbf{n} \cdot \mathbf{u} = \tilde{u}_{||}\,, \qquad \mathbf{n} \times \left(\mathbf{t}^{(n)} \times \mathbf{n}\right) = \mathbf{t}^{(n)} - \mathbf{n}\left(\mathbf{t}^{(n)} \cdot \mathbf{n}\right) = \tilde{\mathbf{t}}_{\perp} \qquad \forall \mathbf{x} \in A_{m2}\,. \tag{5.61}$$

are the two mixed boundary conditions.

We remark that equation $((5.60)_2)$ $[(5.61)_1]$ remains valid for the velocity field as well provided that the velocity is prescribed (in the tangent plane of A_{m1}) [parallel to the normal on A_{m2}].

When solving dynamical problems the knowledge of boundary conditions is not sufficient to make the equations governing the problem determinate. At the beginning of the motion (in the initial configuration) one should also prescribe the displacement field and the velocity field. Let us denote the prescribed displacement and velocity fields by $\mathbf{u}_o$ and $\mathbf{v}_o$ at $t = t_o = 0$, that is, at the beginning of the motion. Then

$$\mathbf{u}(\mathbf{X}, t)|_{t=t_o=0} = \mathbf{u}_o \quad \text{and} \quad \dot{\mathbf{u}}(\mathbf{X}, t)|_{t=t_o=0} = \mathbf{v}_o \qquad \forall \mathbf{X} \in V^{\circ} \tag{5.62}$$

are the initial conditions. We remark that the compatibility conditions

$$\mathbf{u}_o = \tilde{\mathbf{u}}|_{t=t_o=0} \quad \text{and} \quad \mathbf{v}_o = \dot{\tilde{\mathbf{u}}}|_{t=t_o=0} \qquad \forall \mathbf{X} \in A_u^{\circ} \tag{5.63}$$

should also be satisfied. They express that the initial conditions and the displacement boundary conditions should be consistent with each other.

5.5. Problems

PROBLEM 5.1: Given the matrix of the Cauchy stress tensor at the point P of the current configuration of the body. The unit normal to a plane which passes through the point P is denoted by $\mathbf{n}$:

$$\underline{\mathbf{t}} = \begin{bmatrix} 58.4 & 0.0 & -28.8 \\ 0.0 & -40.0 & 0.0 \\ -28.8 & 0.0 & 41.6 \end{bmatrix} [\text{MPa}]\,, \qquad \mathbf{n} = 0.7, \mathbf{i}_1 + 0.1, \mathbf{i}_2 + \frac{\sqrt{2}}{2}\,\mathbf{e}_3\,.$$

Determine (a) the principal stresses and principal directions and (b) the normal stress $\sigma^{(n)}$ as well as the shearing stresses $\tau^{(n)}$ acting on the plane with normal $\mathbf{n}$.

PROBLEM 5.2: Consider a circular cylinder of radius R and assume that the axis of the cylinder coincides with the coordinate axis x_3. The Cauchy stresses in the cylinder are given by

$$t_{13} = t_{31} = -\mu\vartheta x_2\,, \qquad t_{23} = t_{32} = \mu\vartheta x_1\,,$$
$$t_{11} = t_{22} = t_{33} = t_{12} = t_{21} = 0\,,$$

where μ is a positive constant and ϑ is constant. Show that the outer surface of the cylinder is stress free.

PROBLEM 5.3: Given the deformation of a continuum

$$\mathbf{x} = aX_1\mathbf{i}_1 - bX_2\mathbf{i}_2 + cX_3\mathbf{i}_3\,,$$

where a, b and c are non zero constants. Assume that the Cauchy stress tensor is known:

$$[t_{k\ell}] = \begin{bmatrix} t_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [t_{k\ell}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [t_{k\ell}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t_{33} \end{bmatrix}$$

where $t_{11} = t_{22} = t_{33} = \sigma_\circ = \text{constant}$. Find the first and second Piola-Kirchoff stress tensors for each $[t_{k\ell}]$.

PROBLEM 5.4: Given the matrix of the Cauchy stress tensor in spatial description:

$$\underline{\mathbf{t}} = \begin{bmatrix} 0 & 0 & ax_2 \\ 0 & 0 & -bx_3 \\ ax_2 & -bx_3 & 0 \end{bmatrix}$$

Assume that the stresses are considered at the point P with coordinates $x_1 = 0$, $x_2 = b^2$ and $x_3 = a$. Find (a) the three scalar invariants, (b) the principal stresses and principal directions and then (c) determine the maximum shear stress and the normal to the plane on which it acts.

PROBLEM 5.5: Let $\mathbf{n}$ and $\hat{\mathbf{n}}$ be unit normals to two different surface elements at the point P in the current configuration of the body. The stress vectors on these surface elements are denoted by $\mathbf{t}^{(n)}$ and $\mathbf{t}^{(\hat{n})}$. Prove that the relation

$$\mathbf{n} \cdot \mathbf{t}^{(n)} = \hat{\mathbf{n}} \cdot \mathbf{t}^{(\hat{n})}$$

holds if and only if the Cauchy stress tensor is symmetric.

PROBLEM 5.6: Prove the following equalities:

$$\boldsymbol{S} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T}\,, \qquad S_{AB} = F^{-1}_{A\ell}\tau_{\ell k}F^{-1}_{kB}\,, \tag{5.64}$$

$$\boldsymbol{T} = \frac{1}{2}\left(\boldsymbol{R}^T \cdot \boldsymbol{P} + \boldsymbol{P}^T \cdot \boldsymbol{R}\right)\,, \qquad T_{AB} = \frac{1}{2}\left(R_{Ak}P_{kB} + P_{Ak}R_{kB}\right)\,. \tag{5.65}$$

CHAPTER 6

Fundamental laws for deformable bodies

6.1. Principle of mass conservation

6.1.1. Mass and density. The mass of the body is a measure of the material quantity that is in the body. The mass is a positive quantity. Consider now body $\boldsymbol{\mathcal{B}}$ in the current configuration. Let Δm be the mass in a volume element ΔV. The mass distribution within the body is characterized by the limit

$$\rho(\mathbf{x}, t) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{\mathrm{d}m}{\mathrm{d}V} \tag{6.1}$$

which is called density. We shall assume that the density ρ is a continuous (or a piecewise) continuous function of the location at a point of time t.

Remark 6.1: We used this concept when we introduced the concept of body forces for a unit mass in Subsection 5.1.1.

With the density

$$m = \int_V \rho \,\mathrm{d}V \tag{6.2}$$

is the total mass of the body. Equations (6.1) and (6.2) are related to the current configuration. In the initial configuration

$$\rho^\circ = \lim_{\Delta V^\circ \to 0} \frac{\Delta m^\circ}{\Delta V^\circ} = \frac{\mathrm{d}m^\circ}{\mathrm{d}V^\circ} \qquad \text{and} \qquad m^\circ = \int_{V^\circ} \rho^\circ \,\mathrm{d}V^\circ \tag{6.3}$$

are the density and the total mass of the body.

It follows from their definitions that the densities ρ° and ρ are positive quantities.

6.1.2. Principle of mass conservation in global and local forms. The body is said to be closed if no material is added to it (or removed from it) during the motion. In the sequel we shall assume that the body we consider is closed.

It is a fundamental law of Newton's mechanics that the mass can not be transformed into energy. It is therefore an invariant, i.e., a constant quantity:

$$m^\circ = m\,. \tag{6.4}$$

This principle is that of the mass conservation in global form.

Let $V^{\circ\prime}$ be a subregion of the volume region V° occupied by the body in the initial configuration. In a limit case we allow equality, i.e., $V^{\circ\prime} \subseteq V^\circ$. The boundary surface of $V^{\circ\prime}$ is denoted by $A^{\circ\prime}$. The outward unit normal on $A^{\circ\prime}$ is denoted by $\mathbf{n}^\circ$. During the motion subregion $V^{\circ\prime}$ of the body is deformed into subregion $V' \subseteq V$ of the current configuration. Subregion V' is bounded by the surface A' with outward unit normal $\mathbf{n}$. Figure 6.1 shows among others, the subregion V' and its surface A'.

According to the principle of mass conservation

$$\int_{V^{\circ\prime}} \rho^\circ \, \mathrm{d}V^\circ = \int_{V'} \rho \, \mathrm{d}V = \mathit{constant}\,.$$

Hence

$$\left(\int_{V'} \rho \, \mathrm{d}V\right)^{\boldsymbol{\cdot}} = \int_{V'} (\rho)^{\boldsymbol{\cdot}} \, \mathrm{d}V + \int_{V'} \rho \, (\mathrm{d}V)^{\boldsymbol{\cdot}} = \int_{V'} \left[(\rho)^{\boldsymbol{\cdot}} + \rho \, (\mathbf{v} \cdot \nabla)\right] \mathrm{d}V = 0\,,$$

where we have taken relation (3.35) also into account. Since V' is arbitrary it follows that the integrand should vanish:

$$\boxed{(\rho)^{\boldsymbol{\cdot}} + \rho \, (\mathbf{v} \cdot \nabla) = (\rho)^{\boldsymbol{\cdot}} + \rho \, v_{\ell,\ell} = 0\,.} \tag{6.5a}$$

This equation is the principle of mass conservation in local form. We can rewrite it into other form if we utilize formula (3.57) set up for the material time derivative of a scalar field:

$$\boxed{\frac{\partial \rho}{\partial t} + (\rho \mathbf{v}) \cdot \nabla = \frac{\partial \rho}{\partial t} + (\rho \, v_\ell)_{,\ell} = 0\,.} \tag{6.5b}$$

Equations (6.5) are also referred to as continuity equations in spatial description.

According to the principle of mass conservation the masses in the volume elements $\mathrm{d}V^\circ$ and $\mathrm{d}V$ are the same, therefore, it holds that

$$\mathrm{d}m = \rho^\circ \, \mathrm{d}V^\circ = \rho \, \mathrm{d}V \tag{6.6a}$$

from where we get

$$(\mathrm{d}m)^{\boldsymbol{\cdot}} = (\rho^\circ \, \mathrm{d}V^\circ)^{\boldsymbol{\cdot}} = (\rho \, \mathrm{d}V)^{\boldsymbol{\cdot}} = 0\,. \tag{6.6b}$$

Upon substitution of equation (2.94) into (6.6a) we have

$$\rho = \frac{\mathrm{d}V^\circ}{\mathrm{d}V} \rho^\circ = \frac{1}{J} \rho^\circ \qquad \text{or} \qquad J = \frac{\rho^\circ}{\rho} \tag{6.7}$$

which proves that the Jacobian is a positive quantity. This conclusion is in accordance with Remark 2.2.

Equation

$$\boxed{\rho J = \rho^\circ} \tag{6.8}$$

is the continuity equation in material (Lagrangian) description.

In the linear theory of deformations a comparison of equations (4.28) and (6.7) shows that $\rho^\circ = \rho$.

6.2. Balance laws

6.2.1. External and effective forces. Two systems of forces are said to be statically equivalent if the moment vector spaces that belong to the two force systems are the same. A necessary and sufficient condition for the two force systems to be equivalent is that the resultants and the moment resultants (they should be taken about a fixed point for example about the origin) are the same. In this section we shall clarify what are the necessary and sufficient conditions for the dynamic equilibrium.

Subregion V' of the body $\boldsymbol{\mathcal{B}}$ – see Figure 6.1 for details – is subjected to body forces $\rho\mathbf{b}$ and surface forces $\mathbf{t}^{(n)}$ exerted on A' by the part of body $\boldsymbol{\mathcal{B}}$ outside V'. These force systems constitute the system of external forces acting on the part of body $\boldsymbol{\mathcal{B}}$ in V'. We shall denote the system of external forces simply by $\left[\rho\mathbf{b},\mathbf{t}^{(n)}\right]$. The density of the system of effective forces on V' is $\rho\mathbf{a}$ where $\mathbf{a}$ is the acceleration. The system of effective forces is denoted by $[\rho\mathbf{a}]$.

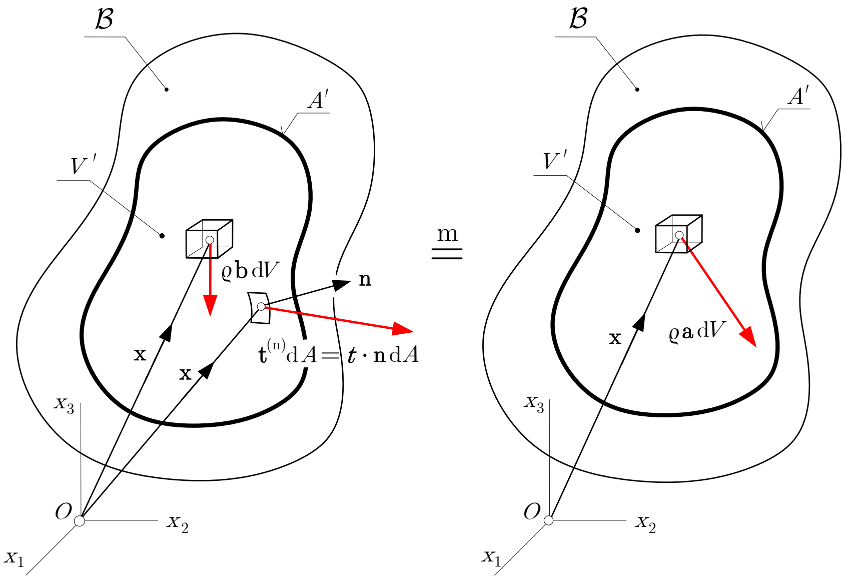

FIGURE 6.1. Free body diagram for the inner part of the body in V'

The system of external forces acting on the part of body $\boldsymbol{\mathcal{B}}$ in V' should be statically equivalent to the system of effective forces acting also on this part of the body:

$$\left[\rho\mathbf{b},\mathbf{t}^{(n)}\right] \overset{m}{=} [\rho\mathbf{a}] \tag{6.9}$$

This condition is the fundamental law of dynamics.

We remark that the letter m over the equality sign expresses that the moment vector spaces are the same for the two force systems considered.

6.2.2. Equation of motion.

6.2.2.1. *Equation of motion in spatial description.* Figure 6.1 is a free body diagram for the subregion V' of the body. It shows both the external forces and the effective forces. Since they are statically equivalent the resultants have to be the same. Consequently, it holds that

$$\int_{V'} \rho \mathbf{b}\,\mathrm{d}V + \int_{A'} \mathbf{t}^{(n)}\,\mathrm{d}A = \int_{V'} \rho \mathbf{a}\,\mathrm{d}V\,. \tag{6.10}$$

If we now substitute (5.14) for $\mathbf{t}^{(n)}$ we get

$$\int_{V'} \rho \mathbf{b}\,\mathrm{d}V + \int_{A'} \boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A = \int_{V'} \rho \mathbf{a}\,\mathrm{d}V\,. \tag{6.11}$$

We can now apply the divergence theorem (1.179) to transform the surface integral into a volume integral. After a rearrangement we have

$$\int_{V'} (\boldsymbol{t}\cdot\nabla + \rho\mathbf{b} - \rho\mathbf{a})\;\mathrm{d}V = \mathbf{0}\,. \tag{6.12}$$

Since the subregion V' is arbitrary it follows that the integrand should vanish:

$$\boxed{\boldsymbol{t}\cdot\nabla + \rho\mathbf{b} = \rho\mathbf{a}\,, \qquad t_{k\ell,\ell} + \rho\, b_k = \rho\, a_k\,.} \tag{6.13}$$

This equation is the equation of motion in spatial description.

6.2.2.2. *Equation of motion in material description.* In the initial configuration the volume region V' and its boundary surface A' are denoted by $V^{\circ\prime}$ and $A^{\circ\prime}$. The outward unit normal on $A^{\circ\prime}$ is $\mathbf{n}^{\circ}$. The body forces for a unit mass are given by

$$\mathbf{b}^{\circ} = \mathbf{b}\,[\boldsymbol{\chi}\,(\mathbf{X};t)] \tag{6.14}$$

which shows that $\mathbf{b}^{\circ}$ is regarded as if it were a function of the material coordinates. If attached to the point P° the acceleration is denoted by

$$\mathbf{a}^{\circ} = \mathbf{a}\,[\boldsymbol{\chi}\,(\mathbf{X};t)] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{u}^{\circ} = \ddot{\mathbf{u}}^{\circ}\,. \tag{6.15}$$

According to the principle of mass conservation $\rho^{\circ}\mathrm{d}V^{\circ} = \rho\,\mathrm{d}V$. It is also worth recalling that $\mathbf{n}\,\mathrm{d}A = J\boldsymbol{F}^{-T}\cdot\mathbf{n}^{\circ}\,\mathrm{d}A^{\circ}$. On the basis of all that has been said in this Subsection so far we can rewrite equation (6.11) into the following form:

$$\int_{V^{\circ\prime}} \rho^{\circ}\,\mathbf{b}^{\circ}\,\mathrm{d}V^{\circ} + \int_{A^{\circ\prime}} J\,\boldsymbol{t}\cdot\boldsymbol{F}^{-T}\cdot\mathbf{n}^{\circ}\,\mathrm{d}A^{\circ} = \int_{V^{\circ\prime}} \rho^{\circ}\,\mathbf{a}^{\circ}\,\mathrm{d}V\,, \tag{6.16}$$

where in view of (5.24) and (5.27)

$$J\,\boldsymbol{t}\cdot\boldsymbol{F}^{-T} = \boldsymbol{P} = \boldsymbol{F}\cdot\left(J\,\boldsymbol{F}^{-1}\cdot\boldsymbol{t}\cdot\boldsymbol{F}^{-T}\right) = \boldsymbol{F}\cdot\boldsymbol{S}\,.$$

Hence

$$\int_{A^{\circ\prime}} J\,\boldsymbol{t}\cdot\boldsymbol{F}^{-T}\cdot\mathbf{n}^{\circ}\,\mathrm{d}A^{\circ} = \int_{A^{\circ\prime}} \boldsymbol{P}\cdot\mathbf{n}^{\circ}\,\mathrm{d}A^{\circ} = \int_{A^{\circ\prime}} \boldsymbol{F}\cdot\boldsymbol{S}\cdot\mathbf{n}^{\circ}\,\mathrm{d}A^{\circ}$$

is the form of the surface integral in (6.16). After transforming it into a volume integral by utilizing the divergence theorem we get

$$\int_{A^{\circ\prime}} J\,\boldsymbol{t}\cdot\boldsymbol{F}^{-T}\cdot\mathbf{n}^\circ\,\mathrm{d}A^\circ = \int_{A^{\circ\prime}} \boldsymbol{F}\cdot\boldsymbol{S}\cdot\mathbf{n}^\circ\,\mathrm{d}A^\circ = \int_{V^{\circ\prime}} (\boldsymbol{F}\cdot\boldsymbol{S})\cdot\nabla^\circ\,\mathrm{d}V^\circ\,.$$

Inserting this result into (6.16) yields

$$\int_{V^{\circ\prime}} [(\boldsymbol{F}\cdot\boldsymbol{S})\cdot\nabla^\circ + \rho^\circ\,\mathbf{b}^\circ - \rho^\circ\,\mathbf{a}^\circ]\,\mathrm{d}V^\circ = \mathbf{0} \tag{6.17}$$

from where, because of the arbitrariness of the volume region $V^{\circ\prime}$, it follows that

$$\boxed{(\boldsymbol{F}\cdot\boldsymbol{S})\cdot\nabla^\circ + \rho^\circ\,\mathbf{b}^\circ = \rho^\circ\,\mathbf{a}^\circ\,.} \tag{6.18}$$

This equation is the equation of motion in material description.

6.2.3. Symmetry of the Cauchy stress tensor. It also follows from the conditions of equivalence that the moments of the external and effective forces about the origin should be the same:

$$\int_{V'} \mathbf{x}\times\rho\mathbf{b}\,\mathrm{d}V + \int_{A'} \mathbf{x}\times\mathbf{t}^{(n)}\,\mathrm{d}A = \int_{V'} \mathbf{x}\times\rho\mathbf{a}\,\mathrm{d}V\,.$$

Substitute again (5.14) for $\mathbf{t}^{(n)}$. We obtain

$$\int_{V'} \mathbf{x}\times\rho\mathbf{b}\,\mathrm{d}V + \int_{A'} \mathbf{x}\times\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A = \int_{V'} \mathbf{x}\times\rho\mathbf{a}\,\mathrm{d}V\,, \tag{6.19}$$

where the surface integral can be transformed into a volume integral by the use of the divergence theorem (1.179). We obtain

$$\int_{A'} \mathbf{x}\times\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A = \int_{V'} (\mathbf{x}\times\boldsymbol{t})\cdot\nabla\,\mathrm{d}V\,, \tag{6.20}$$

in which the integrand in the volume integral can be manipulated into a more suitable form by performing, where possible, the derivation:

$$(\mathbf{x}\times\boldsymbol{t})\cdot\nabla = \overset{\downarrow}{\mathbf{x}}\times\boldsymbol{t}\cdot\frac{\partial}{\partial x_\ell}\,\mathbf{i}_\ell + \mathbf{x}\times(\boldsymbol{t}\cdot\nabla) = \underbrace{\frac{\partial\mathbf{x}}{\partial x_\ell}}_{\mathbf{i}_\ell}\times\underbrace{\boldsymbol{t}\cdot\mathbf{i}_\ell}_{\mathbf{t}_\ell} + \mathbf{x}\times(\boldsymbol{t}\cdot\nabla) =$$

$$= 2\underbrace{\left(-\frac{1}{2}\mathbf{t}_\ell\times\mathbf{i}_\ell\right)}_{\mathbf{t}^a} + \mathbf{x}\times(\boldsymbol{t}\cdot\nabla) = 2\mathbf{t}^a + \mathbf{x}\times(\boldsymbol{t}\cdot\nabla)\,. \tag{6.21}$$

Here $\mathbf{t}^a$ is the axial vector of the Cauchy stress tensor. Substitute (6.21) into the right side of (6.20) and then the result into (6.19). After a rearrangement we have

$$\int_{V'} \Big[2\mathbf{t}^a + \mathbf{x}\times\underbrace{(\boldsymbol{t}\cdot\nabla + \rho\mathbf{b} - \rho\mathbf{a})}_{=\mathbf{0}\ \ \text{(compare to (6.13))}}\Big]\,\mathrm{d}V = 2\int_{V'} \mathbf{t}^a\,\mathrm{d}V = \mathbf{0}\,.$$

Since V' is arbitrary it follows that

$$\boxed{\mathbf{t}^a = \mathbf{0}\,.\ \text{Hence}\ \ \boldsymbol{t} = \boldsymbol{t}^T\,.} \tag{6.22}$$

In words: The Cauchy stress tensor is a symmetric tensor.

On the basis of equations (5.24) and (5.27) we may conclude that the first Piola-Kirchhoff stress tensor is not symmetric while the second Piola-Kirchhoff stress tensor is symmetric.

EXERCISE 6.1: Determine the equations of motion in a scalar form in the Cartesian coordinate system (xyz).
It follows from a comparison of equations (6.13) and (5.12) that

$$\frac{\partial \mathbf{t}_x}{\partial x}+\frac{\partial \mathbf{t}_y}{\partial y}+\frac{\partial \mathbf{t}_z}{\partial z}+\rho\mathbf{b}=\rho\mathbf{a} \tag{6.23}$$

is the vectorial form of the equation of motion in this coordinate system. If we now recall resolution (5.15) of the stress vectors we get the following scalar equations

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+\rho b_x&=\rho a_x\,,\\
\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+\rho b_y&=\rho a_y\,,\\
\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+\rho b_z&=\rho a_z\,.
\end{aligned} \tag{6.24}$$

6.2.4. Equilibrium equations. A reference frame is a model of a rigid body from where we observe the motion of other bodies. We may attach various coordinate systems to the rigid body mentioned, we shall, however, regard one of these coordinate systems as the reference frame – see Subsection 8.2.2.1 for more details in this respect. The reference frame is an inertial reference frame if the body on which it is defined is at rest, i.e., there are no forces acting on it. When devising the equation of motion and the symmetry condition for the Cauchy-stress tensor we assumed tacitly that the moving body is considered in an inertial reference frame.

If we assume that $\mathbf{a}=\mathbf{0}$ we arrive at an equilibrium problem by which we mean that the load is applied gradually and the body gets into the current configuration via a series of equilibrium configurations. If the acceleration is not negligible we shall write $\mathbf{f}^{(a)}$ for $\rho\mathbf{b}-\rho\mathbf{a}$ – see equation (7.28b) in Section 7.3.4 – by attacking the dynamic problem considered as if it were an equilibrium problem. However, we can apply this simplification only if the acceleration field in the body is, ab ovo, known.

For equilibrium problems we can drop the term $\rho\mathbf{a}$ in the equation of motion:

$$\boldsymbol{t}\cdot\nabla+\rho\mathbf{b}=\mathbf{0}\,,\qquad t_{k\ell,\ell}+\rho\, b_k=0\,. \tag{6.25a}$$

This equation is called equilibrium equation. For simplicity it is worth introducing the notation

$$\rho\mathbf{b}=\mathbf{f}\,,\qquad\qquad \rho b_k=f_k \tag{6.25b}$$

where $\mathbf{f}$ is the body force per unit volume.

EXERCISE 6.2: Consider a body assuming infinitesimal deformations. Then there is no difference between the Cauchy stress tensor and the second Piola-Kirchhoff stress tensor which are known for the body considered:

$$\underset{(3\times3)}{\underline{\mathbf{t}}} = \underset{(3\times3)}{\underline{\mathbf{S}}} = \underset{(3\times3)}{\underline{\boldsymbol{\sigma}}} = \begin{bmatrix} a_1X_1^2 & 0 & a_1X_1X_2 \\ 0 & a_2X_3 & a_2X_2 \\ a_1X_1X_2 & a_2X_2 & a_2X_3 \end{bmatrix}.$$

where a_1 and a_2 are non zero constants. Find the body forces $\mathbf{f}$ required for the body to be in equilibrium.

Note first that the symmetry condition is satisfied. Rewriting equations $(6.25)_2$ in the form

$$f_k = -t_{kl,l}$$

we get

$$f_1 = \left(\frac{\partial\sigma_{11}}{\partial X_1} + \frac{\partial\sigma_{12}}{\partial X_2} + \frac{\partial\sigma_{13}}{\partial X_3}\right) = -2a_1X_1\,,$$

$$f_2 = \left(\frac{\partial\sigma_{21}}{\partial X_1} + \frac{\partial\sigma_{22}}{\partial X_2} + \frac{\partial\sigma_{23}}{\partial X_3}\right) = 0\,,$$

$$f_3 = \left(\frac{\partial\sigma_{31}}{\partial X_1} + \frac{\partial\sigma_{32}}{\partial X_2} + \frac{\partial\sigma_{33}}{\partial X_3}\right) = -a_1X_1 - 2a_2$$

which means that

$$\mathbf{f} = -2a_1X_1\,\mathbf{i}_1 - (a_1X_1 + 2a_2)\,\mathbf{i}_3\,.$$

EXERCISE 6.3: Given the stress state of the beam shown in Figure 6.2 by the following relations:

$$\sigma_{11} = \frac{3\mathfrak{f}\ell^2}{2b^3}\left(1 - \frac{4x_1^2}{\ell^2}\right)x_3, \qquad \sigma_{22} = 0\,, \qquad \sigma_{33} = \frac{3\mathfrak{f}}{2b}\left(x_3 - \frac{4x_3^3}{3b^2} + \frac{b}{3}\right),$$

$$\sigma_{12} = \sigma_{21} = 0\,, \qquad \sigma_{23} = \sigma_{32} = 0\,, \quad \sigma_{31} = \sigma_{13} = -\frac{3\mathfrak{f}x_1}{2b}\left(1 - \frac{4x_3^2}{b^2}\right).$$

Though there are no body forces the stresses depend on a parameter $\mathfrak{f}$. Are the equilibrium equations satisfied? What is the load on the lateral faces of the beam in terms of $\mathfrak{f}$?

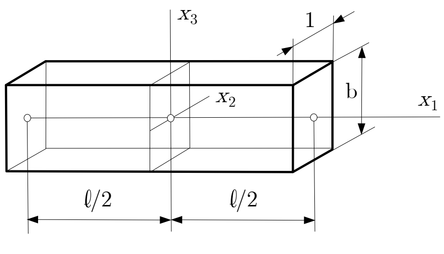

FIGURE 6.2. Beam with uniform rectangular cross section

Making use of the derivatives

$$\frac{\partial\sigma_{11}}{\partial X_1} = -\frac{12\mathfrak{f}X_1X_3}{b^3}, \qquad \frac{\partial\sigma_{22}}{\partial X_2} = 0\,, \qquad \frac{\partial\sigma_{33}}{\partial X_3} = \frac{3\mathfrak{f}}{2b}\left(1 - \frac{4X_3^2}{b^2}\right),$$
$$\frac{\partial\sigma_{12}}{\partial X_1} = \frac{\partial\sigma_{12}}{\partial X_2} = \frac{\partial\sigma_{23}}{\partial X_2} = \frac{\partial\sigma_{23}}{\partial X_3} = 0\,, \qquad \frac{\partial\sigma_{31}}{\partial X_1} = -\frac{3\mathfrak{f}}{2b}\left(1 - \frac{4X_3^2}{b^2}\right),$$
$$\frac{\partial\sigma_{13}}{\partial X_3} = \frac{12\mathfrak{f}X_1X_3}{b^3}$$

we get

$$\frac{\partial\sigma_{11}}{\partial X_1} + \frac{\partial\sigma_{12}}{\partial X_2} + \frac{\partial\sigma_{13}}{\partial X_3} = -\frac{12\mathfrak{f}X_1X_3}{b^3} + \frac{12\mathfrak{f}X_1X_3}{b^3} = 0\,,$$
$$\frac{\partial\sigma_{21}}{\partial X_1} + \frac{\partial\sigma_{22}}{\partial X_2} + \frac{\partial\sigma_{23}}{\partial X_3} \equiv 0\,,$$
$$\frac{\partial\sigma_{31}}{\partial X_1} + \frac{\partial\sigma_{32}}{\partial X_2} + \frac{\partial\sigma_{33}}{\partial X_3} = -\frac{3\mathfrak{f}}{2b}\left(1 - \frac{4X_3^2}{b^2}\right) + \frac{3\mathfrak{f}}{2b}\left(1 - \frac{4X_3^2}{b^2}\right) = 0$$

which shows that the equilibrium equations are satisfied.

On the lateral faces $x_2 = -1/2$ and $x_2 = 1/2$ the stress vectors vanish: $\mathbf{t}^{(-x_2)} = \mathbf{t}^{(x_2)} = \mathbf{0}$. These faces are, therefore, free of load.

On the bottom face $x_3 = -b/2$ we have:

$$-\,\mathbf{t}^{(-x_3)} = -\left(\sigma_{13}\,\mathbf{i}_1 + \sigma_{23}\,\mathbf{i}_2 + \sigma_{33}\,\mathbf{i}_3\right)|_{x_3=-b/2} =$$
$$= \frac{3\mathfrak{f}x_1}{2b}\,(1-1)\,\mathbf{i}_1 - \frac{3\mathfrak{f}}{2b}\left(-\frac{b}{2} + \frac{b}{6} + \frac{b}{3}\right)\mathbf{i}_3 = \mathbf{0}\,.$$

This face is also free of load.

On the top face $x_3 = b/2$ we obtain similarly:

$$\mathbf{t}^{(x_3)} = \left(\sigma_{13}\,\mathbf{i}_1 + \sigma_{23}\,\mathbf{i}_2 + \sigma_{33}\,\mathbf{i}_3\right)|_{x_3=b/2} =$$
$$= -\frac{3\mathfrak{f}x_1}{2b}\,(1-1)\,\mathbf{i}_1 + \frac{3\mathfrak{f}}{2b}\left(\frac{b}{2} - \frac{b}{6} + \frac{b}{3}\right)\mathbf{i}_3 = \mathfrak{f}\,\mathbf{i}_3\,.$$

This result shows that the beam is subjected to a constant surface load on the top face of the beam.

6.2.5. Equations in cylindrical coordinate systems. Figure 6.3 shows a stress element in the cylindrical coordinate system $(R\vartheta z)$ by using the notations of Exercise 5.2. With these notations we can give the stress tensor

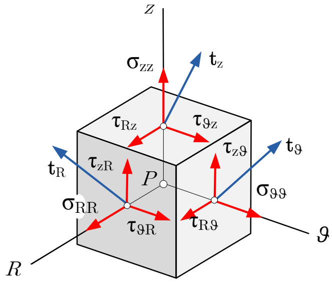

FIGURE 6.3. Cubic stress element in the cylindrical coordinate system $(R\vartheta z)$

and its matrix in the same way as we did in Exercise 5.2:

$$\boldsymbol{t} = \mathbf{t}_R \circ \mathbf{i}_R + \mathbf{t}_\vartheta \circ \mathbf{i}_\vartheta + \mathbf{t}_z \circ \mathbf{i}_z \,,$$

$$\underset{(3\times3)}{\underline{\boldsymbol{t}}} = \left[\begin{array}{c|c|c} \underset{(3\times3)}{\underline{\mathbf{t}_R}} & \underset{(3\times3)}{\underline{\mathbf{t}_\vartheta}} & \underset{(3\times3)}{\underline{\mathbf{t}_z}} \end{array} \right] = \begin{bmatrix} \sigma_{RR} & \tau_{R\vartheta} & \tau_{Rz} \\ \tau_{\vartheta R} & \sigma_{\vartheta\vartheta} & \tau_{\vartheta z} \\ \tau_{zR} & \tau_{z\vartheta} & \sigma_{zz} \end{bmatrix}. \tag{6.26}$$

Assume that $\boldsymbol{\sigma}$ is a function of the cylindrical coordinates. Then for the divergence of $\boldsymbol{\sigma}$ we can write by detailing the calculations that

$$\begin{aligned}
\boldsymbol{t} \cdot \nabla = \underset{(1.195)}{\uparrow} &= (\mathbf{t}_R \circ \mathbf{i}_R + \mathbf{t}_\vartheta \circ \mathbf{i}_\vartheta + \mathbf{t}_z \circ \mathbf{i}_z) \cdot \left(\frac{\partial}{\partial R}\mathbf{i}_R + \frac{1}{R}\frac{\partial}{\partial \vartheta}\mathbf{i}_\vartheta + \frac{\partial}{\partial z}\mathbf{i}_z \right) = \\
&= \frac{\partial \mathbf{t}_R}{\partial R} \underbrace{\mathbf{i}_R \cdot \mathbf{i}_R}_{=1} + \frac{\partial \mathbf{t}_\vartheta}{\partial R} \underbrace{\mathbf{i}_\vartheta \cdot \mathbf{i}_R}_{=0} + \frac{\partial \mathbf{t}_z}{\partial R} \underbrace{\mathbf{i}_z \cdot \mathbf{i}_R}_{=0} + \\
&+ \frac{\partial \mathbf{t}_R}{R\partial \vartheta} \underbrace{\mathbf{i}_R \cdot \mathbf{i}_\vartheta}_{=0} + \frac{\mathbf{t}_R}{R} \underbrace{\frac{\mathrm{d}\mathbf{i}_r}{\mathrm{d}\vartheta} \cdot \mathbf{i}_\vartheta}_{=1} + \frac{\partial \mathbf{t}_\vartheta}{R\partial \vartheta} \underbrace{\mathbf{i}_\vartheta \cdot \mathbf{i}_\vartheta}_{=1} + \frac{\mathbf{t}_\vartheta}{R} \underbrace{\frac{\mathrm{d}\mathbf{i}_\vartheta}{\mathrm{d}\vartheta} \cdot \mathbf{i}_\vartheta}_{=0} + \frac{\partial \mathbf{t}_z}{R\partial \vartheta} \underbrace{\mathbf{i}_z \cdot \mathbf{i}_\vartheta}_{=0} + \\
&+ \frac{\partial \mathbf{t}_R}{\partial z} \underbrace{\mathbf{i}_R \cdot \mathbf{i}_z}_{=0} + \frac{\partial \mathbf{t}_\vartheta}{\partial z} \underbrace{\mathbf{i}_\vartheta \cdot \mathbf{i}_z}_{=0} + \frac{\partial \mathbf{t}_z}{\partial z} \underbrace{\mathbf{i}_z \cdot \mathbf{i}_z}_{=1} = \\
&= \frac{\partial \mathbf{t}_R}{\partial R} + \frac{\mathbf{t}_R}{R} + \frac{\partial \mathbf{t}_\vartheta}{R\partial \vartheta} + \frac{\partial \mathbf{t}_z}{\partial z} \,.
\end{aligned} \tag{6.27}$$

Upon substitution of the divergence $\boldsymbol{\sigma} \cdot \nabla$ into (6.13) we obtain the equation of motion in a symbolic form in terms of the stress vectors $\mathbf{t}_R$, $\mathbf{t}_\vartheta$ and $\mathbf{t}_z$:

$$\frac{\partial \mathbf{t}_R}{\partial R} + \frac{\mathbf{t}_R}{R} + \frac{1}{R}\frac{\partial \mathbf{t}_\vartheta}{\partial \vartheta} + \frac{\partial \mathbf{t}_z}{\partial z} + \rho \mathbf{b} = \rho \mathbf{a} \,. \tag{6.28}$$

Making use of the derivative

$$\frac{1}{R}\frac{\partial \mathbf{t}_\vartheta}{\partial \vartheta} = \left(\frac{1}{R}\frac{\partial \tau_{R\vartheta}}{\partial \vartheta} - \frac{\sigma_{\vartheta\vartheta}}{R} \right) \mathbf{i}_R + \frac{1}{R}\left(\frac{\partial \sigma_{\vartheta\vartheta}}{\partial \vartheta} + \tau_{R\vartheta} \right) \mathbf{i}_\vartheta + \frac{1}{R}\frac{\tau_{z\vartheta}}{\partial \vartheta}\, \mathbf{i}_\vartheta$$

we can also set up the scalar equations of motion:

$$\left.\begin{aligned} \frac{\partial\sigma_{RR}}{\partial R}+\frac{\sigma_{RR}-\sigma_{\vartheta\vartheta}}{R}+\frac{1}{R}\frac{\partial\tau_{R\vartheta}}{\partial\vartheta}+\frac{\partial\tau_{Rz}}{\partial z}+\rho b_R=\rho a_R\,,\\ \frac{\partial\tau_{\vartheta R}}{\partial R}+\frac{\tau_{\vartheta R}+\tau_{R\vartheta}}{R}+\frac{1}{R}\frac{\partial\sigma_{\vartheta\vartheta}}{\partial\vartheta}+\frac{\partial\tau_{\vartheta z}}{\partial z}+\rho b_\vartheta=\rho a_\vartheta\,,\\ \frac{\partial\tau_{zR}}{\partial R}+\frac{\tau_{zR}}{R}+\frac{1}{R}\frac{\partial\tau_{z\vartheta}}{\partial\vartheta}+\frac{\partial\sigma_{zz}}{\partial z}+\rho b_z=\rho a_z\,. \end{aligned}\right\} \tag{6.29}$$

For equilibrium problems $a_R = a_\vartheta = a_z = 0$.

EXERCISE 6.4: We speak about axisymmetric problems if

$$\begin{aligned} &\sigma_{RR}=\sigma_{RR}(R,z)\,, \quad \sigma_{\vartheta\vartheta}=\sigma_{\vartheta\vartheta}(R,z)\,, \quad \sigma_{zz}=\sigma_{zz}(R,z)\,,\\ &\tau_{zR}=\tau_{Rz}(R,z)\,, \quad \tau_{R\vartheta}=\tau_{\vartheta R}=0\,, \quad \tau_{\vartheta z}=\tau_{z\vartheta}=0\,. \end{aligned}$$

What are then the scalar equilibrium equations? (Keep in mind that $b_\vartheta = 0$ for axisymmetric problems!)

We obtain the following equilibrium equations from (6.29):

$$\left.\begin{aligned} \frac{\partial\sigma_{RR}}{\partial R}+\frac{\sigma_{RR}-\sigma_{\vartheta\vartheta}}{R}+\frac{\partial\tau_{Rz}}{\partial z}+\rho b_R=0\,,\\ \frac{\partial\tau_{zR}}{\partial R}+\frac{\tau_{zR}}{R}+\frac{\partial\sigma_{zz}}{\partial z}+\rho b_z=0\,. \end{aligned}\right\} \tag{6.30}$$

6.3. Energy theorem

6.3.1. Energy theorem in spatial description. Dot multiplying the equation of motion (6.13) by the velocity field $\mathbf{v}$ yields

$$\rho\,\mathbf{v}\cdot\mathbf{a}=\mathbf{v}\cdot(\boldsymbol{t}\cdot\nabla)+\rho\,\mathbf{v}\cdot\mathbf{b}\,, \qquad \rho\,v_k\,a_k=v_k\,t_{k\ell,\ell}+\rho\,v_k\,b_k \tag{6.31}$$

in which

$$\rho\,\mathbf{v}\cdot\mathbf{a}=\rho\,\mathbf{v}\cdot(\mathbf{v})^{\bullet}=\frac{1}{2}\,\rho\,(\mathbf{v}\cdot\mathbf{v})^{\bullet}=\frac{1}{2}\,\rho\,\left(\mathbf{v}^2\right)^{\bullet} \tag{6.32}$$

and

$$\mathbf{v}\cdot(\overset{\downarrow}{\boldsymbol{t}}\cdot\nabla)=(\mathbf{v}\cdot\boldsymbol{t})\cdot\nabla-\overset{\downarrow}{\mathbf{v}}\cdot\boldsymbol{t}\cdot\nabla\,. \tag{6.33}$$

If we take relations $(1.96)_3$, (3.6b) and (3.11) into account we can rewrite the above equation into the following form

$$\begin{aligned} \mathbf{v}\cdot(\overset{\downarrow}{\boldsymbol{t}}\cdot\nabla)&=(\mathbf{v}\cdot\boldsymbol{t})\cdot\nabla-\boldsymbol{t}\cdot\cdot(\mathbf{v}\circ\nabla)=(\mathbf{v}\cdot\boldsymbol{t})\cdot\nabla-\boldsymbol{t}\cdot\cdot\boldsymbol{l}=\\ &=(\mathbf{v}\cdot\boldsymbol{t})\cdot\nabla-\boldsymbol{t}\cdot\cdot\boldsymbol{d}-\underbrace{\boldsymbol{t}\cdot\cdot\boldsymbol{\Omega}}_{=0}=(\mathbf{v}\cdot\boldsymbol{t})\cdot\nabla-\boldsymbol{t}\cdot\cdot\boldsymbol{d}\,, \end{aligned} \tag{6.34}$$

where the energy product $\boldsymbol{t}\cdot\cdot\boldsymbol{\Omega}$ vanishes since $\boldsymbol{t}$ is symmetric and $\boldsymbol{\Omega}$ is skew. Let us now substitute (6.32) and (6.34) back into (6.31) and integrate the result over the volume V. We get

$$\frac{1}{2}\int_V\rho\,\left(\mathbf{v}^2\right)^{\bullet}\,\mathrm{d}V=\int_V(\mathbf{v}\cdot\boldsymbol{t})\cdot\nabla\,\mathrm{d}V-\int_V\boldsymbol{t}\cdot\cdot\boldsymbol{d}\,\mathrm{d}V+\int_V\rho\,\mathbf{v}\cdot\mathbf{b}\,\mathrm{d}V\,. \tag{6.35}$$

Note that the first volume integral on the right side can be transformed into a surface integral by using the divergence theorem. Thus

$$\frac{1}{2}\int_V \rho\,\left(\mathbf{v}^2\right)^{\boldsymbol{\cdot}}\,\mathrm{d}V = \int_A \mathbf{v}\cdot\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A + \int_V \rho\,\mathbf{v}\cdot\mathbf{b}\,\mathrm{d}V - \int_V \boldsymbol{t}\cdot\cdot\,\boldsymbol{d}\,\mathrm{d}V\,. \qquad (6.36)$$

This equation includes the following quantities:

- The time derivative of kinetic energy of the continuum:

$$(\mathcal{K})^{\boldsymbol{\cdot}} = \left(\frac{1}{2}\int_V \rho\,\mathbf{v}^2\,\mathrm{d}V\right)^{\boldsymbol{\cdot}} = \frac{1}{2}\int_V \rho\,\left(\mathbf{v}^2\right)^{\boldsymbol{\cdot}}\,\mathrm{d}V\,. \qquad (6.37\text{a})$$

  (Because of the mass conservation $(\rho\,\mathrm{d}V)^{\boldsymbol{\cdot}} = 0$!)
- The power of the external forces, i.e., that of the surface tractions $\mathbf{t}^{(n)}$ and body forces $\rho\mathbf{b}$:

$$P_{\text{ext}} = \int_A \mathbf{v}\cdot\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A + \int_V \rho\,\mathbf{v}\cdot\mathbf{b}\,\mathrm{d}V\,. \qquad (6.37\text{b})$$

- The stress power per unit volume in the current configuration (stress power density):

$$\phi_T = \boldsymbol{t}\cdot\cdot\,\boldsymbol{d}\,. \qquad (6.37\text{c})$$

- The power of the internal forces:

$$P_{\text{int}} = -\int_V \phi_T\,\mathrm{d}V = -\int_V \boldsymbol{t}\cdot\cdot\,\boldsymbol{d}\,\mathrm{d}V\,. \qquad (6.37\text{d})$$

Consequently, equation

$$\boxed{(\mathcal{K})^{\boldsymbol{\cdot}} = P_{\text{ext}} + P_{\text{int}}\,,} \qquad (6.38)$$

which follows from (6.36), is the global form of the energy theorem.

Remark 6.2: For rigid bodies $P_{\mathsf{int}} = 0$, therefore, the energy theorem simplifies:

$$(\mathcal{K})^{\boldsymbol{\cdot}} = P_{\mathsf{ext}}\,. \qquad (6.39)$$

Note that equation (6.35) holds not only for the whole body, which occupies the volume region V, but for any parts of the body in the volume region V' – see Figure 6.1. Hence

$$\boxed{\begin{aligned} \frac{1}{2}\rho\,\left(v^2\right)^{\boldsymbol{\cdot}} + \phi_T &= (\mathbf{v}\cdot\boldsymbol{t})\cdot\nabla + \rho\,\mathbf{v}\cdot\mathbf{b}\,,\\ \frac{1}{2}\rho\,\left(v^2\right)^{\boldsymbol{\cdot}} + \underbrace{t_{k\ell}\,d_{k\ell}}_{\phi_T} &= (v_p\,t_{pq})_{,q} + \rho\,v_p\,b_p\,, \end{aligned}} \qquad (6.40)$$

which is the local form of the energy theorem.

6.3.2. Energy theorem in material description. Since $\mathbf{v}(\mathbf{x};t) = \mathbf{v}^\circ(\mathbf{X};t)$ and $\rho\,\mathrm{d}V = \rho^\circ \mathrm{d}V^\circ$ it follows on the basis of (6.37a) that

$$(\mathcal{K})^\cdot = \left(\frac{1}{2}\int_{V^\circ} \rho^\circ\ (\mathbf{v}^\circ)^2\ \mathrm{d}V^\circ\right)^\cdot = \frac{1}{2}\int_{V^\circ} \rho\ (\mathbf{v}^o\cdot\mathbf{v}^\circ)^\cdot\ \mathrm{d}V^\circ = \int_{V^\circ} \rho^\circ\,\mathbf{v}^\circ\cdot\mathbf{a}^\circ\,\mathrm{d}V^\circ \tag{6.41}$$

By taking equation $(5.29)_2$ and then the mass conservation low again into account for the power of external forces we get

$$P_{\mathrm{ext}} = \int_{A^\circ} \mathbf{v}^\circ\cdot\boldsymbol{F}\cdot\boldsymbol{S}\cdot\mathbf{n}^\circ\,\mathrm{d}A^\circ + \int_{V^\circ} \rho^\circ\,\mathbf{v}^\circ\cdot\mathbf{b}^\circ\,\mathrm{d}V^\circ\,. \tag{6.42}$$

As regards the integral

$$P_{\mathrm{int}} = -\int_V \boldsymbol{t}\cdot\cdot\,\boldsymbol{d}\,\mathrm{d}V$$

it follows from

(a) equation (5.27) that

$$\boldsymbol{t} = \frac{1}{J}\boldsymbol{F}\cdot\boldsymbol{S}\cdot\boldsymbol{F}^T\,, \qquad t_{k\ell} = \frac{1}{J}F_{kA}S_{AB}F_{B\ell}\,. \tag{6.43a}$$

(c) According to (3.50b) it also holds that

$$\boldsymbol{d} = \boldsymbol{F}^{-T}\cdot(\boldsymbol{E})^\cdot\cdot\boldsymbol{F}^{-1}\,, \qquad d_{k\ell} = F^{-1}_{kM}\,(E_{MN})^\cdot\,F^{-1}_{N\ell}\,. \tag{6.43b}$$

Making use of equations (6.43) we can manipulate the energy product $\boldsymbol{t}\cdot\cdot\,\boldsymbol{d}\,\mathrm{d}V$ into a more suitable form:

$$\begin{aligned}\boldsymbol{t}\cdot\cdot\,\boldsymbol{d}\,\mathrm{d}V &= t_{k\ell}d_{kl}\mathrm{d}V = \frac{1}{J}F_{kA}S_{AB}F_{B\ell}\,F^{-1}_{kM}\,(E_{MN})^\cdot\,F^{-1}_{N\ell}\,\underbrace{J\,\mathrm{d}V^\circ}_{\mathrm{d}V} = \\ &= F_{\ell A}S_{AB}F_{Bk}\,F^{-1}_{kM}\,(E_{MN})^\cdot\,F^{-1}_{N\ell}\,\mathrm{d}V^\circ = S_{AB}\,\underbrace{F^{-1}_{N\ell}F_{\ell A}}_{\delta_{NA}}\,\underbrace{F_{Bk}F^{-1}_{kM}}_{\delta_{BM}}\,(E_{MN})^\cdot\,\mathrm{d}V^\circ = \\ &= S_{AB}\,(E_{AB})^\cdot\,\mathrm{d}V^\circ = \boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^\cdot\,\mathrm{d}V^\circ\,,\end{aligned} \tag{6.44}$$

where we have taken into account that $t_{k\ell} = t_{\ell k}$. Hence

$$P_{\mathrm{int}} = -\int_V \boldsymbol{t}\cdot\cdot\,\boldsymbol{d}\,\mathrm{d}V = -\int_{V^\circ} \boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^\cdot\,\mathrm{d}V^\circ \tag{6.45}$$

A comparison of equations (6.38), (6.41), (6.42) and (6.45) yields

$$\begin{aligned}&\frac{1}{2}\int_{V^\circ}\rho\,\left((\mathbf{v}^o)^2\right)^\cdot\,\mathrm{d}V^\circ = \\ &\quad = \int_{A^\circ} \mathbf{v}^\circ\cdot\boldsymbol{F}\cdot\boldsymbol{S}\cdot\mathbf{n}^\circ\,\mathrm{d}A^\circ + \int_{V^\circ}\rho^\circ\,\mathbf{v}^\circ\cdot\mathbf{b}^\circ\,\mathrm{d}V^\circ - \int_{V^\circ}\boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^\cdot\,\mathrm{d}V^\circ\end{aligned} \tag{6.46a}$$

or if we apply the divergence theorem

$$\begin{aligned}&\frac{1}{2}\int_{V^\circ}\rho\,\left((\mathbf{v}^o)^2\right)^\cdot\,\mathrm{d}V^\circ + \int_{V^\circ}\boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^\cdot\,\mathrm{d}V^\circ = \\ &\quad = \int_{V^\circ}(\mathbf{v}^\circ\cdot\boldsymbol{F}\cdot\boldsymbol{S})\cdot\nabla^\circ\,\mathrm{d}V^\circ + \int_{V^\circ}\rho^\circ\,\mathbf{v}^\circ\cdot\mathbf{b}^\circ\,\mathrm{d}V^\circ\,.\end{aligned} \tag{6.46b}$$

Consequently,

$$\boxed{\frac{1}{2}\rho\left((\mathbf{v}^{o})^{2}\right)^{\bullet}+\boldsymbol{S}\cdot\cdot(\boldsymbol{E})^{\bullet}=(\mathbf{v}^{\circ}\cdot\boldsymbol{F}\cdot\boldsymbol{S})\cdot\nabla^{\circ}+\rho^{\circ}\,\mathbf{v}^{\circ}\cdot\mathbf{b}^{\circ}\,.} \tag{6.47}$$

This equation is the local form of the energy theorem in material description.

REMARK 6.3: It is worthy of mention that the energy theorem is not a fundamental law of continuum mechanics since we have derived it from the equation of motion by performing appropriate mathematical transformations.

6.4. The first theorem of thermodynamics

6.4.1. The first theorem of thermodynamics in spatial description. First we shall introduce some new quantities:

- Assume that there exists an *internal energy per unit mass* (specific internal energy) within the body. We shall denote it by $e=e(\mathbf{x};t)$. With e the *total internal energy* $\mathcal{E}$ within the subregion V' of V is given by
$$\mathcal{E}=\int_{V'}\rho e\,\mathrm{d}V\,. \tag{6.48a}$$
- The internal heat generation in a unit mass per unit time (heat source distribution) is denoted by $h=h(\mathbf{x};t)$ (possibly from a phase change or transmission of electric current). It is obvious that the heat produced (or subtracted) in subregion (from subregion) V' is
$$\mathcal{Q}_V=\int_{V'}\rho h\,\mathrm{d}V\,. \tag{6.48b}$$
- The heat flux vector $\mathbf{q}$ is the heat energy that passes through a unit surface perpendicular to $\mathbf{q}$ per unit time. With $\mathbf{q}$ the heat flow across the surface element $\mathrm{d}\mathbf{A}=\mathbf{n}\mathrm{d}A$ is given by $-\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}A$. The heat input (or output) of the subregion via its surface A' can, therefore, be calculated as
$$\mathcal{Q}_A=-\int_{A'}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}A=-\int_{V'}\mathbf{q}\cdot\nabla\,\mathrm{d}V\,. \tag{6.48c}$$
- Hence, the total heat input (or output) is
$$P_Q=\mathcal{Q}_V+\mathcal{Q}_A=\int_{V'}\rho h\,\mathrm{d}V-\int_{A'}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}A=\int_{V'}(\rho h-\mathbf{q}\cdot\nabla)\,\mathrm{d}V\,. \tag{6.48d}$$
- The kinetic energy of the subregion is
$$\mathcal{K}=\frac{1}{2}\int_{V'}\rho\,\mathbf{v}^2\,\mathrm{d}V\,, \tag{6.48e}$$
where $\mathbf{v}(\mathbf{x};t)$ is the velocity field of the body in spatial description.
- On the basis of (6.37b)
$$P_{\mathrm{ext}}=\int_{V'}\mathbf{v}\cdot\rho\mathbf{b}\,\mathrm{d}V+\int_{A'}\mathbf{v}\cdot\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A\,. \tag{6.48f}$$
is the power of the external forces acting on the subregion V'.

The first theorem of thermodynamics[1] states that the time-rate of change of the total energy is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time [13]. The total energy is the sum of the kinetic energy and the internal energy. The principle of energy conservation can, therefore, be expressed as

$$\boxed{(\mathcal{K}+\mathcal{E})^{\cdot} = P_{\text{ext}} + P_Q\,.} \tag{6.49}$$

Upon substitution of equations (6.48e), (6.48a), (6.48f) and (6.48d) we have

$$\begin{aligned}\left(\frac{1}{2}\int_{V'} \rho\,\mathbf{v}^2\,\mathrm{d}V + \int_{V'} \rho\, e\,\mathrm{d}V\right)^{\cdot} &= \\ &= \int_{V'} \mathbf{v}\cdot\rho\mathbf{b}\,\mathrm{d}V + \int_{A'} \mathbf{v}\cdot\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A + \int_{V'} (\rho h - \mathbf{q}\cdot\nabla)\,\mathrm{d}V\,,\end{aligned} \tag{6.50}$$

where

$$\left(\frac{1}{2}\int_{V'} \rho\,\mathbf{v}^2\,\mathrm{d}V\right)^{\cdot} = \int_{V'} \rho\,\mathbf{v}\cdot\mathbf{a}\,\mathrm{d}V + \frac{1}{2}\int_{V'} \mathbf{v}^2\,\underbrace{(\rho\,\mathrm{d}V)^{\cdot}}_{=0}\,, \tag{6.51a}$$

$$\left(\int_{V'} \rho e\,\mathrm{d}V\right)^{\cdot} = \int_{V'} \rho\,(e)^{\cdot}\,\mathrm{d}V + \frac{1}{2}\int_{V'} e\,\underbrace{(\rho\,\mathrm{d}V)^{\cdot}}_{=0}\,, \tag{6.51b}$$

(conservation of mass) and

$$\int_{A'} \mathbf{v}\cdot\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A = \int_{V'} (\mathbf{v}\cdot\boldsymbol{t})\cdot\nabla\,\mathrm{d}V = \int_{V'} \mathbf{v}\cdot(\boldsymbol{t}\cdot\nabla)\,\mathrm{d}V + \int_{V'} \overset{\downarrow}{\mathbf{v}}\cdot\boldsymbol{t}\cdot\nabla\,\mathrm{d}V\,.$$

As regards the last integral in this equation we can utilize $(1.96)_3$, (3.6b) and (3.11) to manipulate its integrand into a more suitable form:

$$\overset{\downarrow}{\mathbf{v}}\cdot\boldsymbol{t}\cdot\nabla = \boldsymbol{t}\cdot\cdot(\mathbf{v}\circ\nabla) = \boldsymbol{t}\cdot\cdot\boldsymbol{l} = \boldsymbol{t}\cdot\cdot\boldsymbol{d} + \underbrace{\boldsymbol{t}\cdot\cdot\boldsymbol{\Omega}}_{=0} = \boldsymbol{t}\cdot\cdot\boldsymbol{d}\,. \tag{6.51c}$$

Hence

$$\int_{A'} \mathbf{v}\cdot\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A = \int_{V'} \mathbf{v}\cdot(\boldsymbol{t}\cdot\nabla)\,\mathrm{d}V + \int_{V'} \boldsymbol{t}\cdot\cdot\boldsymbol{d}\,\mathrm{d}V\,. \tag{6.51d}$$

Substitute equations (6.51a), (6.51b) and (6.51d) into equation (6.49). A subsequent rearrangement yields

$$\int_{V'} \left[\mathbf{v}\cdot\underbrace{(\boldsymbol{t}\cdot\nabla + \rho\mathbf{b} - \rho\mathbf{a})}_{=\mathbf{0}} - \left(\rho\frac{\mathrm{D}e}{\mathrm{D}t} - \boldsymbol{t}\cdot\cdot\boldsymbol{d} - \rho h + \mathbf{q}\cdot\nabla\right)\right]\mathrm{d}V = 0 \tag{6.52}$$

Since V' is arbitrary it follows that

$$\boxed{\rho\,(e)^{\cdot} = \underbrace{\boldsymbol{t}\cdot\cdot\boldsymbol{d}}_{\phi_T} + \underbrace{\rho h - \mathbf{q}\cdot\nabla}_{\phi_Q}\,.} \tag{6.53}$$

[1] Rudolf Julius Clausius, 1822-1888

where ϕ_T and ϕ_Q are the stress power density (or simply stress power) and heat power per unit volume (or simply heat power) in the current configuration. For rigid body motions the strain rate tensor $\boldsymbol{d}$ is zero tensor, therefore, the stress power is also zero.

Equation (6.53) is the local form of the first law of thermodynamics in spatial description.

In words: the time rate of the specific internal energy is the sum of the stress power density and the heat power per unit volume.

6.4.2. The first theorem of thermodynamics in material description. With the mass conservation $\rho \mathrm{d}V = \rho^\circ \mathrm{d}V^\circ$ the total internal energy takes the form:

$$\mathcal{E} = \int_{V'} \rho\, e\, \mathrm{d}V = \int_{V^{\circ\prime}} \rho^\circ e^\circ\, \mathrm{d}V^\circ\,, \tag{6.54}$$

where $e^\circ(\mathbf{X};t) = e[\boldsymbol{\chi}(\mathbf{X};t)\,;t] = e(\mathbf{x};t)$. We get in the same way from (6.48b) that

$$\mathcal{Q}_V = \int_{V'} \rho h\, \mathrm{d}V = \int_{V^{\circ\prime}} \rho^\circ h^\circ\, \mathrm{d}V^\circ \tag{6.55}$$

in which $h^\circ(\mathbf{X};t) = h[\boldsymbol{\chi}(\mathbf{X};t)\,;t] = h(\mathbf{x};t)$.

By using (2.89) we can rewrite the integral (6.48c) into the following form

$$\mathcal{Q}_A = -\int_{A'} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}A = -\int_{A^{\circ\prime}} J\boldsymbol{F}^{-T}\cdot\mathbf{q}\cdot\mathbf{n}^\circ\,\mathrm{d}A^\circ = \\ = -\int_{A^{\circ\prime}} \mathbf{q}^\circ\cdot\mathbf{n}^\circ\,\mathrm{d}A^\circ = -\int_{V^{\circ\prime}} \mathbf{q}^\circ\cdot\nabla^\circ\,\mathrm{d}V^\circ\,, \tag{6.56}$$

where

$$\mathbf{q}^\circ = J\boldsymbol{F}^{-T}\cdot\mathbf{q} \tag{6.57}$$

is the heat flux in the initial configuration. Consequently,

$$P_Q = \mathcal{Q}_V + \mathcal{Q}_A = \int_{V^{\circ\prime}} \left(\rho^\circ h^\circ - \mathbf{q}^\circ\cdot\nabla^\circ\right)\mathrm{d}V^\circ \tag{6.58}$$

is the total heat input (output) for the body in material description.

Exercise 6.5: Prove that

$$\overset{\downarrow}{\mathbf{v}}{}^\circ\cdot\boldsymbol{F}\cdot\boldsymbol{S}\cdot\nabla^\circ = J\,\boldsymbol{d}\cdot\cdot\,\boldsymbol{t} = \boldsymbol{S}\cdot\cdot\left(\boldsymbol{E}\right)^{\boldsymbol{\cdot}} \tag{6.59}$$

Take relations $(2.30)_1$, (5.27) and (3.11b) into account and consider the transformations

$$\overset{\downarrow}{\mathbf{v}}{}^\circ\cdot\boldsymbol{F}\cdot\boldsymbol{S}\cdot\nabla^\circ = \big(\mathbf{v}^\circ\circ\nabla^\circ\big)\cdot\cdot\big(\boldsymbol{F}\cdot\boldsymbol{S}\big) = (v_P\nabla_B)(F_{pA}S_{AB}) = \\ = \underset{v_P=v_p,\ \nabla_B=\nabla_q F_{Bq}}{\uparrow} = \big((v_p\nabla_q)F_{Bq}\big)(F_{pA}S_{AB}) = \underset{S_{AB}=JF^{-1}_{Ak}t_{k\ell}F^{-1}_{\ell B}}{\uparrow} = \\ = (v_{p,q})\underbrace{F^{-1}_{\ell B}F_{Bq}}_{\delta_{\ell q}}\underbrace{F_{pA}F^{-1}_{Ak}}_{\delta_{pk}}J\,t_{k\ell} = \underset{v_{k,\ell}=d_{k\ell}+\Omega_{k\ell}}{\uparrow} = Jd_{k\ell}\,t_{k\ell} + J\underbrace{\Omega_{k\ell}\,t_{k\ell}}_{=0} = \\ = J\,\boldsymbol{d}\cdot\cdot\,\boldsymbol{t} = \underset{(6.44),\,(2.94)}{\uparrow} = \boldsymbol{S}\cdot\cdot\left(\boldsymbol{E}\right)^{\boldsymbol{\cdot}}\,.$$

which prove our statement.

Upon substitution of (6.41), (6.54), (6.42) and (6.58) into (6.49) we have

$$\int_{V^{\circ\prime}} \rho\,\mathbf{v}^{o}\cdot\mathbf{a}^{\circ}\,\mathrm{d}V^{\circ}+\int_{V^{\circ\prime}}\rho^{\circ}\,(e^{\circ})^{\boldsymbol{\cdot}}\,\mathrm{d}V^{\circ}=$$
$$=\int_{A^{\circ\prime}}\mathbf{v}^{\circ}\cdot\boldsymbol{F}\cdot\boldsymbol{S}\cdot\mathbf{n}^{\circ}\,\mathrm{d}A^{\circ}+\int_{V^{\circ\prime}}\rho^{\circ}\,\mathbf{v}^{\circ}\cdot\mathbf{b}^{\circ}\,\mathrm{d}V^{\circ}+\int_{V^{\circ\prime}}(\rho^{\circ}h^{\circ}-\mathbf{q}^{\circ}\cdot\nabla^{\circ})\,\mathrm{d}V^{\circ} \tag{6.60}$$

in which

$$\int_{A^{\circ\prime}}\mathbf{v}^{\circ}\cdot\boldsymbol{F}\cdot\boldsymbol{S}\cdot\mathbf{n}^{\circ}\,\mathrm{d}A^{\circ}=\int_{V^{\circ\prime}}\big(\mathbf{v}^{\circ}\cdot\boldsymbol{F}\cdot\boldsymbol{S}\big)\cdot\nabla^{\circ}\,\mathrm{d}V^{\circ}=$$
$$=\int_{V^{\circ\prime}}\overset{\downarrow}{\mathbf{v}}{}^{\circ}\cdot\boldsymbol{F}\cdot\boldsymbol{S}\cdot\nabla^{\circ}\,\mathrm{d}V^{\circ}+\int_{V^{\circ\prime}}\mathbf{v}^{\circ}\cdot\big((\boldsymbol{F}\cdot\boldsymbol{S})\cdot\nabla^{\circ}\big)\,\mathrm{d}V^{\circ}=\underset{(6.59)}{\uparrow}=$$
$$=\int_{V^{\circ\prime}}\boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^{\boldsymbol{\cdot}}\,\mathrm{d}V^{\circ}+\int_{V^{\circ\prime}}\mathbf{v}^{\circ}\cdot\big((\boldsymbol{F}\cdot\boldsymbol{S})\cdot\nabla^{\circ}\big)\,\mathrm{d}V^{\circ}$$

Rearranging equation (6.60) by utilizing the above relation yields

$$\int_{V^{\circ\prime}}\Big[\mathbf{v}^{\circ}\cdot\underbrace{\big((\boldsymbol{F}\cdot\boldsymbol{S})\cdot\nabla^{\circ}+\rho^{\circ}\mathbf{b}^{\circ}-\rho^{\circ}\mathbf{s}^{\circ}\big)}_{=\mathbf{0}}-$$
$$-\Big(\rho^{\circ}\,(e^{\circ})^{\boldsymbol{\cdot}}-\boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^{\boldsymbol{\cdot}}-\rho^{\circ}h^{\circ}+\mathbf{q}^{\circ}\cdot\nabla^{\circ}\Big)\Big]\,\mathrm{d}V^{\circ}=0 \tag{6.61}$$

from where it follows the local form of the first theorem of thermodynamics in material description:

$$\boxed{\rho^{\circ}\,(e^{\circ})^{\boldsymbol{\cdot}}=\boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^{\boldsymbol{\cdot}}+\rho^{\circ}h^{\circ}-\mathbf{q}^{\circ}\cdot\nabla^{\circ}.} \tag{6.62}$$

6.4.3. A possible form of the constitutive equations. If the temperature is constant (there is no energy dissipation in the form of heat) $h=0$, $\mathbf{q}=\mathbf{0}$. Consequently, equation (6.53) simplifies to the form

$$\boxed{\rho\,(e)^{\boldsymbol{\cdot}}-\boldsymbol{t}\cdot\cdot\,\boldsymbol{d}=0\,.} \tag{6.63}$$

Note that this equation is valid in spatial description only.

If we set h° and $\mathbf{q}^{\circ}$ to zero in equation (6.62) we get (6.63) in material description:

$$\boxed{\rho^{\circ}\,(e^{\circ})^{\boldsymbol{\cdot}}-\boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^{\boldsymbol{\cdot}}=0\,.} \tag{6.64}$$

Remark 6.4: Assume that the body considered has an internal energy function $e^{\circ}=e$. Assume further that the temperature of the body is constant, i.e., there is no heat input or output into or from the body. If the internal energy $e^{\circ}=e$ is a function of the Green-Lagrange strain tensor $\boldsymbol{E}$ only we have

$$(e^{\circ})^{\boldsymbol{\cdot}}=\frac{\partial e^{\circ}}{\partial E_{AB}}\,(E_{AB})^{\boldsymbol{\cdot}}\,. \tag{6.65}$$

With this result we can rewrite equation (6.64) into the following form:

$$\left(\rho^\circ \frac{\partial e^\circ}{\partial E_{AB}} - S_{AB}\right) (E_{AB})^{\boldsymbol{\cdot}} = 0\,. \tag{6.66}$$

Since the previous equation should be fulfilled for any $(E_{AB})^{\boldsymbol{\cdot}}$ it follows that

$$\boxed{\boldsymbol{S} = \rho^\circ \frac{\partial e^\circ}{\partial \boldsymbol{E}}\,, \qquad S_{AB} = \rho^\circ \frac{\partial e^\circ}{\partial E_{AB}}\,.} \tag{6.67}$$

This equation is a possible constitutive equation.

Remark 6.5: An equation similar to (6.67) but valid in the current configuration can also be derived. According to (2.68) it holds that

$$\boldsymbol{e} = \boldsymbol{F}^{-T} \cdot \boldsymbol{E} \cdot \boldsymbol{F}^{-1}\,, \qquad e_{k\ell} = F^{-1}_{kA}\, E_{AB}\, F^{-1}_{B\ell}\,.$$

Hence

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{E}} = \boldsymbol{F}^{-T} \circ \boldsymbol{F}^{-1}\,, \qquad \frac{\partial e_{k\ell}}{\partial E_{AB}} = F^{-1}_{kA}\, F^{-1}_{B\ell}$$

and

$$\frac{\partial e^\circ}{\partial \boldsymbol{E}} \underset{e=e^\circ}{=\uparrow=} \frac{\partial e}{\partial \boldsymbol{e}} \cdot\cdot \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{E}} = \boldsymbol{F}^{-1} \cdot \frac{\partial e}{\partial \boldsymbol{e}} \cdot \boldsymbol{F}^{-T}\,,$$

$$\frac{\partial e^\circ}{\partial E_{AB}} \underset{e=e^\circ}{=\uparrow=} \frac{\partial e}{\partial e_{k\ell}} \frac{\partial e_{k\ell}}{\partial E_{AB}} = F^{-1}_{Ak} \frac{\partial e}{\partial e_{k\ell}} F^{-1}_{\ell B}\,.$$

Making use of the previous equations yields the following partial result:

$$\boldsymbol{F} \cdot \frac{\partial e^\circ}{\partial \boldsymbol{E}} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \left[\boldsymbol{F}^{-1} \cdot \frac{\partial e}{\partial \boldsymbol{e}} \cdot \boldsymbol{F}^{-T}\right] \cdot \boldsymbol{F}^T = \frac{\partial e}{\partial \boldsymbol{e}}\,,$$

$$F_{rA} \frac{\partial e^\circ}{\partial E_{AB}} F_{Bs} = F_{rA} \left[F^{-1}_{Ak} \frac{\partial e}{\partial \partial e_{k\ell}} F^{-1}_{\ell B}\right] F_{Bs} = \frac{\partial e}{\partial e_{k\ell}}\,. \tag{6.68}$$

A comparison of equations (5.27) and (6.7) leads to the following relation:

$$\boldsymbol{t} = \frac{1}{J} \boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^T = \frac{\rho}{\rho^\circ} \boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^T\,. \tag{6.69}$$

Dot multiply equation (6.67) by $\frac{\rho}{\rho^\circ}\boldsymbol{F}$ from left and by $\boldsymbol{F}^T$ from right. If in addition we take relations (6.68) and (6.69) into account we have

$$\boxed{\boldsymbol{t} = \rho \frac{\partial e}{\partial \boldsymbol{e}}\,, \qquad t_{k\ell} = \rho \frac{\partial e}{\partial e_{k\ell}}\,.} \tag{6.70}$$

Here we should assume that the internal energy density e is given in terms of the Euler-Almansi strain tensor $e_{k\ell}$.

6.5. The second theorem of thermodynamics

6.5.1. The second theorem of thermodynamics in spatial description. We shall need the second theorem of thermodynamics[2] [4, 16] if we want to clarify what effect the non-uniform temperature distribution has on the constitutive equations.

Let s be is the entropy per unit mass – the entropy itself is such an elementary thermodynamical quantity which increases if we input heat into the body and decreases if the body loses heat.

Knowing the entropy density we can calculate the total entropy within the part of the body V':

$$S = \int_{V'} \rho s \, \mathrm{d}V \, . \tag{6.71}$$

Since the entropy is related to the heat energy stored in the body it has also a strong relationship with the temperature distribution within the body. (The temperature Θ at a given point of the body is an objective comparative measure which shows if the body is hot or cold at the point considered. On the absolute scale the temperature Θ is a positive quantity $\Theta \geq 0$.)

Relationship

$$\hat{S} = \int_{V'} \frac{\rho h}{\Theta} \, \mathrm{d}V - \int_{A'} \frac{\mathbf{q}}{\Theta} \cdot \mathbf{n} \, \mathrm{d}A \tag{6.72}$$

is the entropy input rate that gives the entropy contribution (increment or decrement) to the part of the body in V' per second. The quotient h/Θ is the entropy source density (or entropy supply density), while the vector $\mathbf{q}/\Theta$ is called entropy flux.

The second theorem of thermodynamics is an axiom which says that the material time derivative of the total entropy is greater than the entropy input rate (equality is possible for reversible processes only):

$$(S)^{\cdot} = \int_{V'} \rho \, (s)^{\cdot} \, \mathrm{d}V \geq \hat{S} = \int_{V'} \frac{\rho h}{\Theta} \, \mathrm{d}V - \int_{A'} \frac{\mathbf{q}}{\Theta} \cdot \mathbf{n} \, \mathrm{d}A \, . \tag{6.73}$$

If we utilize the divergence theorem we can transform the surface integral into a volume integral:

$$\int_{V'} \left[\rho \, (s)^{\cdot} - \frac{\rho h}{\Theta} + \left(\frac{\mathbf{q}}{\Theta} \right) \cdot \nabla \right] \mathrm{d}V \geq 0 \, . \tag{6.74}$$

Since this inequality holds for any subregion V' of the body it follows that

$$\rho \, (s)^{\cdot} \geq \frac{\rho h}{\Theta} - \left(\frac{\mathbf{q}}{\Theta} \right) \cdot \nabla \, , \tag{6.75}$$

or

$$\boxed{\rho \Theta \, (s)^{\cdot} \geq \rho h - \mathbf{q} \cdot \nabla + \frac{\mathbf{q} \cdot (\Theta \nabla)}{\Theta} = \phi_Q + \frac{\mathbf{q} \cdot (\Theta \nabla)}{\Theta} \, ,} \tag{6.76}$$

[2]Sadi Carnot, 1796-1832
Rudolf Julius Emanuel Clausius, 1822-1888

where, according to (6.53), ϕ_Q is the heat power. Inequality (6.76), which is known as Clausius-Duhem inequality[3] (or entropy theorem), is the local form of the second theorem of thermodynamics [16, 31].

We define the dissipation power per unit volume by the following equation:

$$\Phi_D = \rho\Theta\,(s)^{\bullet} - \underbrace{(\rho h - \mathbf{q}\cdot\nabla)}_{\phi_Q=\rho(e)^{\bullet}-\boldsymbol{t}\cdot\cdot\,\boldsymbol{d}} = \rho\Theta\,(s)^{\bullet} - \rho\,(e)^{\bullet} + \boldsymbol{t}\cdot\cdot\,\boldsymbol{d} \geq 0\,. \tag{6.77}$$

It follows from the definition that the dissipation power can not be negative.

It is worthy of mention that relation (6.77) is known as Clausius-Planck inequality[4] [16, 24].

The local form of the entropy theorem, i.e., inequality (6.76) can be given in terms of the dissipation power too:

$$\Phi_D - \frac{\mathbf{q}\cdot(\Theta\nabla)}{\Theta} \geq 0\,. \tag{6.78}$$

If $\Phi_D = 0$ the inequality $\mathbf{q}\cdot(\Theta\nabla) \leq 0$ follows from here. This result is in accordance with the observation that the heat flows from the hotter place towards the cooler place, i.e., in the opposite direction of the temperature gradient.

6.5.2. The second theorem of thermodynamics in material description. Let $s^{\circ}(\mathbf{X};t) = s[\boldsymbol{\chi}(\mathbf{X};t)\,;t] = s(\mathbf{x};t)$ be the entropy distribution in the initial configuration. Making use of this relation

$$S = \int_{V^{\circ\prime}} \rho^{\circ} s^{\circ}\,\mathrm{d}V^{\circ} \tag{6.79}$$

is the total entropy in the part of the body in $V^{\circ\prime}$. With regard to (6.55) it is obvious that

$$\int_{V'} \frac{\rho h}{\Theta}\,\mathrm{d}V = \int_{V^{\circ\prime}} \frac{\rho^{\circ} h^{\circ}}{\Theta}\,\mathrm{d}V^{\circ}\,. \tag{6.80}$$

Recalling (6.56) we may write

$$\int_{A'} \frac{\mathbf{q}}{\Theta}\cdot\mathbf{n}\,\mathrm{d}A = \int_{A^{\circ\prime}} \frac{J\boldsymbol{F}^{-T}\cdot\mathbf{q}}{\Theta}\cdot\mathbf{n}^{\circ}\,\mathrm{d}A^{\circ} = \\ = \int_{A^{\circ\prime}} \frac{\mathbf{q}^{\circ}}{\Theta}\cdot\mathbf{n}^{\circ}\,\mathrm{d}A^{\circ} = \int_{V^{\circ\prime}} \frac{\mathbf{q}^{\circ}}{\Theta}\cdot\nabla^{\circ}\,\mathrm{d}V^{\circ} \tag{6.81}$$

A comparison of (6.79), (6.80) and (6.81) to (6.73) yields

$$\int_{V^{\circ\prime}} \left[\rho^{\circ}\,(s^{\circ})^{\bullet} - \frac{\rho^{\circ} h^{\circ}}{\Theta} + \left(\frac{\mathbf{q}^{\circ}}{\Theta}\right)\cdot\nabla^{\circ}\right]\mathrm{d}V^{\circ} \geq 0\,. \tag{6.82}$$

Hence

$$\rho^{\circ}\,(s^{\circ})^{\bullet} \geq \frac{\rho^{\circ} h^{\circ}}{\Theta} - \left(\frac{\mathbf{q}^{\circ}}{\Theta}\right)\cdot\nabla^{\circ}\,, \tag{6.83}$$

or

$$\boxed{\rho^{\circ}\Theta\,(s^{\circ})^{\bullet} \geq \rho^{\circ} h^{\circ} - \mathbf{q}^{\circ}\cdot\nabla^{\circ} + \frac{\mathbf{q}^{\circ}\cdot(\Theta\nabla^{\circ})}{\Theta}\,.} \tag{6.84}$$

[3]Pierre Duhem, 1861-1916

[4]Max Karl Ernst Ludwig Planck, 1858-1947

It follows from the first theorem of thermodynamics that

$$\rho^\circ \left(e^\circ\right)^\bullet - \boldsymbol{S}\cdot\cdot\left(\boldsymbol{E}\right)^\bullet = \rho^\circ h^\circ - \mathbf{q}^\circ \cdot \nabla^\circ \,. \qquad (6.85)$$

Substituting this relation into (6.84) yields

$$\rho^\circ \Theta \left(s^\circ\right)^\bullet \geq \rho^\circ \left(e^\circ\right)^\bullet - \boldsymbol{S}\cdot\cdot\left(\boldsymbol{E}\right)^\bullet + \frac{\mathbf{q}^\circ \cdot \left(\Theta \nabla^\circ\right)}{\Theta} \,. \qquad (6.86)$$

Equations (6.84) and (6.86) constitute the second theorem of thermodynamics in local form and material description.

6.5.3. Energy conjugate quantities. Assume that the mechanical processes we consider are all isothermal. Then the stress power density can be given in various forms:

$$\phi_T = \boldsymbol{t}\cdot\cdot\boldsymbol{d} = \underset{(5.33)}{\uparrow} = \frac{1}{J}\boldsymbol{\tau}\cdot\cdot\boldsymbol{d} = \underset{(6.59)}{\uparrow} = \frac{1}{J}\boldsymbol{S}\cdot\cdot\left(\boldsymbol{E}\right)^\bullet =$$

$$= \underset{(2.32)}{\uparrow} = \frac{1}{2J}\boldsymbol{S}\cdot\cdot\left(\boldsymbol{C}\right)^\bullet = \underset{(6.64)}{\uparrow} = \frac{1}{J}\rho^\circ \left(e^\circ\right)^\bullet = \underset{(6.8)}{\uparrow} = \rho \left(e\right)^\bullet \,. \qquad (6.87)$$

Utilizing the transformation

$$\boldsymbol{S}\cdot\cdot\left(\boldsymbol{E}\right)^\bullet = \underset{(2.32)}{\uparrow} = \underbrace{\left(\boldsymbol{F}^{-1}\cdot\boldsymbol{P}\right)}_{\boldsymbol{S}}\cdot\cdot\underbrace{\frac{1}{2}\left(\left(\boldsymbol{F}^T\right)^\bullet\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\left(\boldsymbol{F}\right)^\bullet\right)}_{\left(\boldsymbol{E}\right)^\bullet} =$$

$$= \left(\boldsymbol{F}^{-1}\cdot\boldsymbol{P}\right)\cdot\cdot\left(\boldsymbol{F}^T\cdot\left(\boldsymbol{F}\right)^\bullet\right) = \underset{(1.213)}{\uparrow} = \left(\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{P}\right)\cdot\cdot\left(\boldsymbol{F}\right)^\bullet =$$

$$= \boldsymbol{P}\cdot\cdot\left(\boldsymbol{F}\right)^\bullet \qquad (6.88)$$

– $\boldsymbol{F}^{-1}\cdot\boldsymbol{P}$, $\boldsymbol{F}^T$ and $\left(\boldsymbol{F}\right)^\bullet$ correspond to $\boldsymbol{S}$, $\boldsymbol{T}$ and $\boldsymbol{W}$ in (1.213) – leads to the result

$$\phi_T = \frac{1}{J}\boldsymbol{S}\cdot\cdot\left(\boldsymbol{E}\right)^\bullet = \frac{1}{J}\boldsymbol{P}\cdot\cdot\left(\boldsymbol{F}\right)^\bullet \,. \qquad (6.89)$$

Remark 6.6: If we take the time derivative of the product $\boldsymbol{R}^T\cdot\boldsymbol{R}$ we get

$$\left(\boldsymbol{R}^T\right)^\bullet\cdot\boldsymbol{R} = -\boldsymbol{R}^T\cdot\left(\boldsymbol{R}\right)^\bullet = -\left(\left(\boldsymbol{R}^T\right)^\bullet\cdot\boldsymbol{R}\right)^T \qquad (6.90)$$

which shows that the product $\left(\boldsymbol{R}^T\right)^\bullet\cdot\boldsymbol{R}$ is a skew tensor.

We shall also make use of the following manipulations for which equation (6.88) is our point of departure:

$$\boldsymbol{S}\cdot\cdot\left(\boldsymbol{E}\right)^\bullet = \boldsymbol{S}\cdot\cdot\frac{1}{2}\left(\left(\boldsymbol{F}^T\right)^\bullet\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\left(\boldsymbol{F}\right)^\bullet\right) = \boldsymbol{S}\cdot\cdot\left(\left(\boldsymbol{F}^T\right)^\bullet\cdot\boldsymbol{F}\right) =$$

$$= \boldsymbol{S}\cdot\cdot\left(\left(\boldsymbol{U}\cdot\boldsymbol{R}^T\right)^\bullet\cdot\boldsymbol{F}\right) = \boldsymbol{S}\cdot\cdot\left(\left(\boldsymbol{U}\right)^\bullet\cdot\underbrace{\boldsymbol{R}^T\cdot\boldsymbol{R}}_{=\boldsymbol{1}}\cdot\boldsymbol{U} + \boldsymbol{U}\cdot\left(\boldsymbol{R}^T\right)^\bullet\cdot\boldsymbol{R}\cdot\boldsymbol{U}\right) =$$

$$= \boldsymbol{S}\cdot\cdot\left(\left(\boldsymbol{U}\right)^\bullet\cdot\boldsymbol{U}\right) \qquad (6.91)$$

in which on the basis of Remark 6.6 it holds that

$$\boldsymbol{S}\cdot\cdot\left(\boldsymbol{U}\cdot\left(\boldsymbol{R}^T\right)^{\boldsymbol{\cdot}}\cdot\boldsymbol{R}\cdot\boldsymbol{U}\right) = S_{AB}\,U_{AL}\,(R_{Lk})^{\boldsymbol{\cdot}}\,R_{kC}\,U_{CB} =$$
$$= U_{LA}S_{AB}U_{BC}\;(R_{Lk})^{\boldsymbol{\cdot}}\,R_{kC} = \underbrace{(\boldsymbol{U}\cdot\boldsymbol{S}\cdot\boldsymbol{U})}_{\text{symmetric}}\cdot\cdot\underbrace{\left(\left(\boldsymbol{R}^T\right)^{\boldsymbol{\cdot}}\cdot\boldsymbol{R}\right)}_{\text{skew}} = 0\,.$$

On the other hand

$$\boldsymbol{T}\cdot\cdot(\boldsymbol{U})^{\boldsymbol{\cdot}} = \underset{(5.32)}{\uparrow} = \frac{1}{2}((\boldsymbol{U}\cdot\boldsymbol{S}+\boldsymbol{S}\cdot\boldsymbol{U})\cdot\cdot(\boldsymbol{U})^{\boldsymbol{\cdot}} = (\boldsymbol{U})^{\boldsymbol{\cdot}}\cdot\cdot(\boldsymbol{U}\cdot\boldsymbol{S}) =$$
$$= (U_{AB})^{\boldsymbol{\cdot}}\left(U_{AC}S_{CB}\right) = S_{BC}\;(U_{BA})^{\boldsymbol{\cdot}}\,U_{AC} = \boldsymbol{S}\cdot\cdot\left((\boldsymbol{U})^{\boldsymbol{\cdot}}\cdot\boldsymbol{U}\right)\,. \qquad (6.92)$$

Compare (6.91) and(6.91) and take (6.87) into account. We get

$$\phi_T = \frac{1}{J}\,\boldsymbol{T}\cdot\cdot(\boldsymbol{U})^{\boldsymbol{\cdot}} \qquad (6.93)$$

It also follows from (6.87), (6.89) and (6.93) that

$$\rho^\circ\,(e^\circ)^{\boldsymbol{\cdot}} = J\,\boldsymbol{t}\cdot\cdot\boldsymbol{d} = \boldsymbol{\tau}\cdot\cdot\boldsymbol{d} =$$
$$= \boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^{\boldsymbol{\cdot}} = \frac{1}{2}\,\boldsymbol{S}\cdot\cdot(\boldsymbol{C})^{\boldsymbol{\cdot}} = \boldsymbol{T}\cdot\cdot(\boldsymbol{U})^{\boldsymbol{\cdot}} = \boldsymbol{P}\cdot\cdot(\boldsymbol{F})^{\boldsymbol{\cdot}}\,. \qquad (6.94)$$

According to this equation the energy product of a stress tensor and the associated strain rate tensor results in the time rate of the mechanical energy per unit volume in the initial configuration. On the basis of this equation the stress tensors $J\boldsymbol{t}$, $\boldsymbol{\tau}$, $\boldsymbol{S}$, $\boldsymbol{T}$, $\boldsymbol{P}$ and the associated strain rate tensors $\boldsymbol{d}$, $(\boldsymbol{E})^{\boldsymbol{\cdot}}$, $(\boldsymbol{C})^{\boldsymbol{\cdot}}/2$, $(\boldsymbol{U})^{\boldsymbol{\cdot}}$, $(\boldsymbol{F})^{\boldsymbol{\cdot}}$ are called work conjugate pairs or energy conjugate pairs. For further details concerning the work conjugate pairs the reader is referred to [85] – see pages 144-145.

REMARK 6.7: With (6.88) equation (6.62) takes the following form:

$$\rho^\circ\,(e^\circ)^{\boldsymbol{\cdot}} = \boldsymbol{P}\cdot\cdot(\boldsymbol{F})^{\boldsymbol{\cdot}} + \rho^\circ h^\circ - \mathbf{q}^\circ\cdot\nabla^\circ\,. \qquad (6.95)$$

For isothermal deformations the above relationship is simplified to

$$\rho^\circ\,(e^\circ)^{\boldsymbol{\cdot}} = \boldsymbol{P}\cdot\cdot(\boldsymbol{F})^{\boldsymbol{\cdot}} = P_{\ell A}\,(F_{\ell A})^{\boldsymbol{\cdot}}\,. \qquad (6.96)$$

If e° is a differentiable function of the deformation gradient $\boldsymbol{F}$ we obtain

$$(e^\circ)^{\boldsymbol{\cdot}} = \frac{\partial e^\circ}{\partial F_{\ell A}}\,(F_{\ell A})^{\boldsymbol{\cdot}}\,. \qquad (6.97)$$

With this result equation (6.64) can be rewritten into the following form:

$$\left(\rho^\circ\frac{\partial e^\circ}{\partial F_{\ell A}} - P_{\ell A}\right)(F_{\ell A})^{\boldsymbol{\cdot}} = 0\,. \qquad (6.98)$$

This equation should be fulfilled for any $(F_{\ell A})^{\boldsymbol{\cdot}}$. Hence

$$\boxed{\boldsymbol{P} = \rho^\circ\frac{\partial e^\circ}{\partial \boldsymbol{F}}\,, \qquad\qquad P_{\ell A} = \rho^\circ\frac{\partial e^\circ}{\partial F_{\ell A}}\,.} \qquad (6.99)$$

Significance of the constitutive type relations (6.67), (6.70) and (6.99) will be discussed in Section 8.5.5 which is devoted to the material behavior of hyperelastic materials. It is a fundamental issue concerning the isothermal case what quantity the strain energy density e° depends on: $e_{k\ell}$, $F_{\ell A}$ or E_{AB}.

6.6. Problems

PROBLEM 6.1: Given the equilibrium condition of a body:

$$x_1 = (1+\alpha)X_1 + \beta X_2\,, \qquad x_2 = \beta X_1 + (1+\alpha)X_2\,, \quad x_3 = X_3 \tag{6.100}$$

where α and β are constants. Prove that

$$\rho = \frac{\rho^\circ}{(1+\alpha)^2 - \beta^2}\,. \tag{6.101}$$

Are there any restrictions on the constants α and β? If yes give them.

PROBLEM 6.2: Given the velocity field

$$v_1 = \alpha x_1 - \beta x_2, \quad v_2 = \beta x_1 + \alpha x_2, \quad v_3 = \gamma\sqrt{x_1^2 + x_2^2}$$

in which α, β and γ are constants. Find the density in the current configuration provided that ρ° is known. Under what condition can this motion be isochoric?

PROBLEM 6.3: Prove using indicial notation that the Cauchy stress tensor is a symmetric tensor.

PROBLEM 6.4: Assume that the deformations are small. Assume further that we know the stress tensor which is given by its matrix:

$$\underset{(3\times 3)}{\underline{\mathbf{t}}} = \underset{(3\times 3)}{\underline{\boldsymbol{\sigma}}} = \alpha \begin{bmatrix} 6X_1X_3^2 & 0 & -2X_3^3 \\ 0 & 1 & 2 \\ -2X_3^3 & 2 & 3X_1^2 \end{bmatrix} \qquad \alpha = 1\,[\mathrm{N/mm^5}] \tag{6.102}$$

where X_ℓ is measured in mm.

(a) In the absence of body forces the stress field should be self-equilibrated. Check if the stress field (6.102) is really self-equilibrated.

(b) Determine the stress vector at the point $\mathbf{X} = 2\,\mathbf{i}_1 + 3\,\mathbf{i}_2 + 2\,\mathbf{i}_3$ [mm] on the plane $2X_1 + X_2 - X_3 = 5$. (We remark that the unit normal to a plane defined by the equation $a_\ell X_\ell = b$ — a_ℓ and b are constants — is given by the relation $\mathbf{n} = a_\ell\,\mathbf{i}_\ell/\sqrt{a_k a_k}$.)

(c) Determine the normal stress $\sigma^{(n)}$ and the shearing stress $\boldsymbol{\tau}^{(n)}$ at this point on the plane.

PROBLEM 6.5: Assume that the stress tensor within the body is given by the following equation:

$$\underset{(3\times 3)}{\underline{\mathbf{t}}} = a \begin{bmatrix} x_1^2 x_2 & x_1(b^2 - x_2^2) & 0 \\ x_1(b^2 - x_2^2) & \frac{1}{3}x_2(x_2^2 - 3b^2) & 0 \\ 0 & 0 & 2bx_3^2 \end{bmatrix} \tag{6.103}$$

in which a and b are non zero constants. Find the body forces if the stress tensor is equilibrated.

Problem 6.6: Assume that

$$\underset{(3\times 3)}{\underline{\mathbf{t}}} = \mu\alpha \begin{bmatrix} 0 & 0 & \frac{\partial\phi}{\partial x_1} - x_2 \\ 0 & 0 & \frac{\partial\phi}{\partial x_2} + x_1 \\ \frac{\partial\phi}{\partial x_1} - x_2 & \frac{\partial\phi}{\partial x_2} + x_1 & 0 \end{bmatrix}.$$

is the stress field, where μ and α are non zero constants while $\phi\,(x_1, x_2)$ is a harmonic function, i.e., $\Delta\phi = 0$. Does this stress tensor satisfy the equilibrium equations in the absence of body forces?

Problem 6.7: Given the function $f(\mathbf{x}, t)$. Prove that

$$\int_V f(\mathbf{x},t)\mathbf{t}(\mathbf{x},t)\cdot\mathbf{n}\,\mathrm{d}V = \int_A \Big[\mathbf{t}(\mathbf{x},t)\cdot(\nabla f(\mathbf{x},t)) + \rho f(\mathbf{x},t)(\mathbf{b}-\dot{\mathbf{v}})\Big]\,\mathrm{d}A.$$

Problem 6.8: Given the equilibrium configuration of a body and the elements of the Cauchy stress tensor:

$$\begin{aligned} x_1 = 16X_1\,, \qquad x_2 = -\frac{1}{2}X_2\,, \quad x_3 = -\frac{1}{4}X_3 \\ t_{11} = 100\ \mathrm{MPa} \qquad t_{k\ell} = 0 \ \text{ if } \ {}_{k\ell} \neq {}_{11}\,. \end{aligned} \tag{6.104}$$

Determine (a) the first and second Piola-Kirchoff stress tensors then (b) the stress vector $\mathbf{t}^{(n)}$ and the pseudo stress vector $\mathbf{t}^{\circ(n)}$ on a plane with normal $\mathbf{n}^{\circ}$ before deformation.

Problem 6.9: The mass center $\mathbf{x}_c$ a body $\mathcal{B}$ is defined as

$$\mathbf{x}_c = \frac{1}{m}\int_V \rho\mathbf{x}\,\mathrm{d}V.$$

Prove that

$$\frac{D^2\mathbf{x}_c}{Dt^2} = \int_V \rho\mathbf{b}\,\mathrm{d}V + \int_A \boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A.$$

Problem 6.10: Assume that the stress field is self equilibrated. Prove that the average value of the Cauchy stress tensor

$$\bar{t}_{\ell k} = \frac{1}{V}\int_V t_{\ell k}\,\mathrm{d}V$$

can be calculated as

$$\bar{t}_{\ell k} = \frac{1}{2V}\int_V \rho\left(x_k b_\ell + x_\ell b_k\right)\,\mathrm{d}V + \frac{1}{2V}\int_A \left(x_k t_\ell^{(n)} + t_k^{(n)} x_\ell\right)\mathrm{d}A\,.$$

Problem 6.11: Prove the validity of transformation (6.88) using indicial notation.

CHAPTER 7

Energy principles

7.1. Special vector and tensor fields

The special vector and tensor fields satisfy some kinematic equations or fundamental laws of continuum mechanics but not all of them which means that a few other equations of continuum mechanics are, however, not fulfilled.

The velocity field $\hat{\mathbf{v}}$ is said to be *kinematically admissible* (geometrically permissible) if it satisfies the kinematic boundary conditions

$$\hat{\mathbf{v}} = \tilde{\mathbf{v}}, \qquad \hat{v}_k = \tilde{v}_k \qquad \forall \mathbf{x} \in A_v \tag{7.1}$$

and is differentiable up to a required order (at least two times).

{{The strain rate tensor $\hat{\boldsymbol{d}}$ is}} [[The strain rates $\hat{d}_{k\ell}$ are]] said to be kinematically admissible (geometrically permissible) if the kinematic equations

$$\hat{\boldsymbol{d}} = \frac{1}{2}\left(\hat{\mathbf{v}} \circ \nabla + \nabla \circ \hat{\mathbf{v}}\right), \quad \hat{d}_{k\ell} = \frac{1}{2}\left(\hat{v}_{k,\ell} + \hat{v}_{\ell,k}\right) \qquad \forall \mathbf{x} \in V \tag{7.2}$$

have a solution for the velocity field $\hat{\mathbf{v}}$ ($\hat{v}_k$) and the solution satisfies boundary conditions (7.1) imposed on the velocity field.

Conversely, the {{strain rate tensor $\boldsymbol{d}$}} [[strain rates $d_{k\ell}$]] obtained from a given kinematically admissible (geometrically permissible) velocity field $\hat{\mathbf{v}}$ ($\hat{v}_k$) by using kinematic equations (7.2) {{is}} [[are]] also kinematically admissible (geometrically permissible).

The strain rate tensor $\overset{*}{\boldsymbol{d}}$ is compatible if it satisfies the compatibility equation

$$-\nabla \times \overset{*}{\boldsymbol{d}} \times \nabla = \boldsymbol{0}\,, \qquad e_{pqr}e_{sk\ell}\overset{*}{d}_{ps,qk} = 0 \qquad \forall \mathbf{x} \in A_v \tag{7.3}$$

which means that equation

$$\overset{*}{\boldsymbol{d}} = \frac{1}{2}\left(\overset{*}{\mathbf{v}} \circ \nabla + \nabla \circ \overset{*}{\mathbf{v}}\right), \quad \overset{*}{d}_{k\ell} = \frac{1}{2}\left(\overset{*}{v}_{k,\ell} + \overset{*}{v}_{\ell,k}\right) \qquad \forall \mathbf{x} \in V \tag{7.4}$$

has solutions for the velocity field $\overset{*}{\mathbf{v}}$. We remark that the possible velocity fields $\overset{*}{\mathbf{v}}$ may, however, differ from each other in an arbitrary rigid body velocity field.

It is obvious that a kinematically admissible strain rate tensor also satisfies compatibility equations (7.3).

The kinematically admissible displacement field is defined in a similar manner: The displacement field $\hat{\mathbf{u}}$ is *kinematically admissible* (geometrically permissible) if it satisfies the displacement boundary conditions

$$\hat{\mathbf{u}} = \tilde{\mathbf{u}}\,, \qquad \hat{u}_k = \tilde{u}_k\,. \qquad \forall \mathbf{x} \in A_u \tag{7.5}$$

and is differentiable up to a required order (at least two times).

In the linear theory of deformation the kinematically admissible {{strain tensor $\hat{\boldsymbol{\varepsilon}}$ is}} [[strains $\hat{\varepsilon}_{kl}$ are]] said to be kinematically admissible (geometrically permissible) if the kinematic equations

$$\hat{\boldsymbol{\varepsilon}} = \frac{1}{2}\left(\hat{\mathbf{u}}\circ\nabla + \nabla\circ\hat{\mathbf{u}}\right)\,, \quad \hat{\varepsilon}_{k\ell} = \frac{1}{2}\left(\hat{u}_{k,\ell} + \hat{u}_{\ell,k}\right) \qquad \forall \mathbf{x} \in V^\circ = V \tag{7.6}$$

have a solution for the displacement field $\hat{\mathbf{u}}$ ($\hat{u}_k$) and the solution satisfies the displacement boundary conditions (7.5).

In the linear theory of deformation the compatible strain tensor $\overset{*}{\boldsymbol{\varepsilon}}$ fulfills, in accordance with all that has been said in Section 4.60, the compatibility conditions

$$\boldsymbol{\eta} = \nabla\times\overset{*}{\boldsymbol{\varepsilon}}\times\nabla = \boldsymbol{0}\,, \quad \eta_{r\ell} = e_{pqr}e_{sk\ell}\overset{*}{\varepsilon}_{ps,qk} = 0\,, \qquad \forall \mathbf{x} \in V^\circ = V \tag{7.7}$$

which means that equation

$$\overset{*}{\boldsymbol{\varepsilon}} = \frac{1}{2}\left(\overset{*}{\mathbf{u}}\circ\nabla + \nabla\circ\overset{*}{\mathbf{u}}\right)\,, \quad \overset{*}{\varepsilon}_{k\ell} = \frac{1}{2}\left(\overset{*}{u}_{k,\ell} + \overset{*}{u}_{\ell,k}\right) \qquad \forall \mathbf{x} \in V^\circ = V \tag{7.8}$$

has solutions for the displacement field $\overset{*}{\mathbf{u}}$ which, however, may differ from each other in an arbitrary rigid body motion of the continuum.

It is obvious that a kinematically admissible strain tensor also satisfies compatibility condition (7.8).

For given body forces $\rho\mathbf{b} = \mathbf{f}$ the Cauchy stresses [[$\bar{t}_{k\ell}$ are]] {{the Cauchy stress tensor $\bar{\boldsymbol{t}}$ is}} said to be *equilibrated* {*statically admissible*} if [[they satisfy]] {{it satisfies}} the equilibrium equations

$$\begin{aligned} \bar{\boldsymbol{t}}\cdot\nabla + \mathbf{f} = \mathbf{0}\,, &\qquad \bar{t}_{k\ell,\ell} + f_k = 0 \\ \bar{\boldsymbol{t}}^T = \bar{\boldsymbol{t}}\,, &\qquad t_{k\ell} = t_{\ell k} \end{aligned} \qquad \forall \mathbf{x} \in V \tag{7.9a}$$

{and the traction (dynamic) boundary conditions

$$\bar{\boldsymbol{t}}\cdot\mathbf{n} = \tilde{\mathbf{t}}\,, \qquad \bar{t}_{k\ell}n_\ell = \tilde{t}_k\,. \qquad \forall \mathbf{x} \in A_t \tag{7.9b}$$

}

REMARK 7.1: If the acceleration field $\mathbf{a}$ is also given and $\mathbf{f} = \mathbf{f}^{(a)} = \rho\mathbf{b} - \rho\mathbf{a}$ then the Cauchy stresses $\bar{t}_{k\ell}$ are said to be {*dynamically admissible*}.

REMARK 7.2: Conditions (7.1), (7.3), (7.5) the kinematically admissible velocity, displacement and strain fields $\hat{v}$, $\hat{u}$ and $\hat{\varepsilon}$ meet are referred to as side conditions. Conditions (7.9) the statically (dynamically) admissible stress field $\bar{\boldsymbol{t}}$ should satisfy are also called side conditions.

Remark 7.3: The true solutions (or simply the solutions) $\mathbf{v}$, $\mathbf{u}$, $\boldsymbol{\varepsilon}$ (v_k, u_k, $\varepsilon_{k\ell}$) are kinematically admissible. The true solution (or simply solution) $\boldsymbol{t}$ ($t_{k\ell}$) is statically (dynamically) admissible.

The difference between two

kinematically admissible velocity fields kinematically admissible strain rate tensors kinematically admissible displacement fields kinematically admissible strain tensors statically (dynamically) admissible stress tensors	is denoted by	$\delta\mathbf{v}$ (δu_k) $\delta\boldsymbol{d}$ (δu_k) $\delta\mathbf{u}$ (δu_k) $\delta\boldsymbol{\varepsilon}$ ($\delta\varepsilon_{k\ell}$) $\delta\boldsymbol{t}$ ($\delta t_{k\ell}$)

and

is referred to as	virtual velocity field (virtual velocities). virtual strain rate tensor (virtual strain rates). virtual displacement field (virtual displacements). virtual strain tensor (virtual strains). virtual stress tensor (virtual stresses).

It follows from their definition that the virtual quantities should meet the following field equations and boundary conditions:

$$\delta\mathbf{v} = \mathbf{0}\,, \qquad (\delta v_k = 0) \qquad \mathbf{x} \in A_v \tag{7.10a}$$

$$\delta\boldsymbol{d} = \frac{1}{2}\left(\delta\mathbf{v}\circ\nabla + \nabla\circ\delta\mathbf{v}\right)\,, \qquad \left[\delta d_{k\ell} = \frac{1}{2}\left(\delta v_{k,\ell} + \delta v_{\ell,k}\right)\right] \qquad \mathbf{x} \in V \tag{7.10b}$$

$$\delta\mathbf{u} = \mathbf{0}\,, \qquad (\delta u_k = 0) \qquad \mathbf{x} \in A_u \tag{7.10c}$$

$$\delta\boldsymbol{\varepsilon} = \frac{1}{2}\left(\delta\mathbf{u}\circ\nabla + \nabla\circ\delta\mathbf{u}\right)\,, \qquad \left[\delta\varepsilon_{k\ell} = \frac{1}{2}\left(\delta u_{k,\ell} + \delta u_{\ell,k}\right)\right] \qquad \mathbf{x} \in V^\circ = V \tag{7.10d}$$

and

$$\delta\boldsymbol{t}\cdot\nabla = \mathbf{0}\,, \qquad (\delta t_{k\ell,\ell} = 0) \qquad \mathbf{x} \in V \tag{7.11a}$$

$$\delta\boldsymbol{t}^T = \delta\boldsymbol{t}\,. \qquad (\delta t_{k\ell} = \delta t_{\ell k}) \qquad \mathbf{x} \in V \tag{7.11b}$$

$$\delta\boldsymbol{t}\cdot\mathbf{n} = \mathbf{0}\,. \qquad (\delta t_{k\ell} n_\ell = 0) \qquad \mathbf{x} \in A_t \tag{7.11c}$$

The above conditions can again be called side conditions the virtual quantities should meet.

Remark 7.4: The virtual quantities are named variations if the subtrahend is the actual solution and the difference (measured in some norm) between the minuend and subtrahend is sufficiently small.

A bit more rigorous definition for the concept of a virtual quantity is also presented here.

Let us denote the increment of a quantity by the Greek Δ. Assume that the dependency of the quantity considered on other quantities should not be taken into account. Then the virtual quantity is a sufficiently small increment of the quantity considered (for example: $\Delta\mathbf{u}^\circ = \delta\mathbf{u}^\circ$).

Assume now that the quantity in question (for example $\boldsymbol{E}$) is a function of another quantity ($\boldsymbol{E}$ is a function of $\mathbf{u}^\circ$). Then its increment is also a function of the increment of the quantity it depends on. If this is the case the part of its increment linear in the increment of the quantity it depends on is its virtual change. For instance

$$\begin{aligned} &\boldsymbol{E}+\Delta\boldsymbol{E}=\boldsymbol{E}(\mathbf{u}^\circ+\delta\mathbf{u}^\circ)=\\ &=\frac{1}{2}\left[(\mathbf{u}^\circ+\delta\mathbf{u}^\circ)\circ\nabla^\circ+\nabla^\circ\circ(\mathbf{u}^\circ+\delta\mathbf{u}^\circ)+(\nabla^\circ\circ(\mathbf{u}^\circ+\delta\mathbf{u}^\circ))\cdot((\mathbf{u}^\circ+\delta\mathbf{u}^\circ)\circ\nabla^\circ)\right]=\\ &=\boldsymbol{E}+\frac{1}{2}\left[\delta\mathbf{u}^\circ\circ\nabla^\circ+\nabla^\circ\circ\delta\mathbf{u}^\circ+(\nabla^\circ\circ\delta\mathbf{u}^\circ)\cdot(\mathbf{u}^\circ\circ\nabla^\circ)+(\nabla^\circ\circ\mathbf{u}^\circ)\cdot(\delta\mathbf{u}^\circ\circ\nabla^\circ)\right]+\\ &\qquad+\frac{1}{2}(\nabla^\circ\circ\delta\mathbf{u}^\circ)\cdot(\delta\mathbf{u}^\circ\circ\nabla^\circ)\,, \end{aligned}$$

where

$$\begin{aligned} &\delta\boldsymbol{E}=\\ &=\frac{1}{2}\left[\delta\mathbf{u}^\circ\circ\nabla^\circ+\nabla^\circ\circ\delta\mathbf{u}^\circ+(\nabla^\circ\circ\delta\mathbf{u}^\circ)\cdot(\mathbf{u}^\circ\circ\nabla^\circ)+(\nabla^\circ\circ\mathbf{u}^\circ)\cdot(\delta\mathbf{u}^\circ\circ\nabla^\circ)\right] \end{aligned} \tag{7.12}$$

is the virtual Green-Lagrange strain tensor (or the variation of the Green-Lagrange strain tensor). The variation of an arbitrary tensor valued tensor function is defined in the following manner. Let $\boldsymbol{\Psi}(\mathbf{u})$ be an arbitrary tensor function of $\mathbf{u}$. The variation of $\boldsymbol{\Psi}(\mathbf{u})$ with respect to $\mathbf{u}$ is given by the following equation:

$$\delta\boldsymbol{\Psi}(\mathbf{u})=\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\boldsymbol{\Psi}(\mathbf{u}+\varepsilon\delta\mathbf{u})\right|_{\epsilon=0}. \tag{7.13}$$

Note that definition (7.13) yields (7.12) if it is applied to the function $\boldsymbol{E}=\boldsymbol{E}(\mathbf{u}^\circ)$.

7.2. General and complete solution to the equilibrium equations

7.2.1. Solution on volume regions bounded by a single closed surface. The stress tensor $\check{\boldsymbol{t}}$ is equilibrated if it satisfies the equilibrium equation

$$\begin{aligned} &\check{\boldsymbol{t}}\cdot\nabla+\mathbf{f}=\mathbf{0}\,, \qquad \mathbf{f}=\rho\mathbf{b}\,,\\ &\check{t}_{k\ell,\ell}+f_k=0\,, \qquad f_k=\rho b_k \end{aligned} \qquad \forall\mathbf{x}\in V \tag{7.14a}$$

and the symmetry condition

$$\check{\boldsymbol{t}}=\check{\boldsymbol{t}}^T, \qquad \check{t}_{k\ell}=\check{t}_{\ell k}\,, \qquad \forall\mathbf{x}\in V\,. \tag{7.14b}$$

Let $\boldsymbol{\mathcal{H}}$ be a symmetric tensor field on V. We shall call it stress function tensor. Furthermore let $\boldsymbol{t}^{(p)}$ be a particular solution to the equilibrium equation: $\boldsymbol{t}^{(p)}\cdot\nabla+\mathbf{f}=\mathbf{0}$.

Consider now the the stress field

$$\check{\boldsymbol{t}}=-\nabla\times\boldsymbol{\mathcal{H}}\times\nabla+\boldsymbol{t}^{(p)}\,, \qquad \forall\mathbf{x}\in V. \tag{7.15}$$

We shall prove that the first term on the right side, i.e., the stress field $\boldsymbol{t}^\star=-\nabla\times\boldsymbol{\mathcal{H}}\times\nabla$ is the general solution of the homogeneous equilibrium equation $\boldsymbol{t}^\star\cdot\nabla=\mathbf{0}$ on simply connected volume regions V bounded by a single closed surface A. The proof is given in indicial notation.

In the first step we shall show that $\boldsymbol{t}^\star$ is equilibrated.

By taking into account the symmetry of the stress function tensor $\boldsymbol{\mathcal{H}}$, i.e., condition $\mathcal{H}_{yd} = \mathcal{H}_{dy}$ we get that $\boldsymbol{t}^\star$ is also symmetric:

$$t^\star_{p\ell} = \epsilon_{pyk}\epsilon_{\ell dr}\mathcal{H}_{yd,kr} = \epsilon_{\ell dr}\epsilon_{pyk}\mathcal{H}_{yd,kr} = \epsilon_{\ell dr}\epsilon_{pyk}\mathcal{H}_{dy,rk} = t^\star_{\ell p}\,, \qquad \forall\mathbf{x}\in V\,.$$

The homogeneous equilibrium equation $\boldsymbol{t}^\star\cdot\nabla = \mathbf{0}$ is also satisfied since:

$${t^\star}_{p\ell,\ell} = \epsilon_{pyk}\epsilon_{\ell dr}\mathcal{H}_{yd,kr\ell} = 0\,, \qquad \forall\mathbf{x}\in V\,.$$

Here we have taken into account (and shall do the same latter on) that the double dot product of skew and symmetric tensors vanishes. Therefore the product $\epsilon_{\ell dr}\mathcal{H}_{yd,kr\ell} = \epsilon_{dr\ell}\mathcal{H}_{yd,kr\ell}$ is equal to zero since the index pair $_{r\ell}$ is a skew one while the index pair $_{r\ell}$ is symmetric. Hence we have proven that the stress field $\boldsymbol{t}^\star$ is equilibrated.

In the second step we shall prove that $\boldsymbol{t}^\star$ is self-equilibrated.

The stress field $\boldsymbol{t}^\star$ is said to be self-equilibrated if (a) the resultant of the stresses on the boundary surface A vanishes and (b) the moment of the same stresses about a fixed point (say about the origin) also vanishes.

Condition (a) is a surface integral which can be transformed into a volume integral by using the divergence theorem. Since the integrand in the volume integral is the equilibrium equation in the absence of body forces it follows, therefore, that the resultant of the stresses on A is zero:

$$\int_A {t^\star}_{p\ell} n_\ell \,\mathrm{d}A = \int_V {t^\star}_{p\ell,\ell}\,\mathrm{d}V = 0\,.$$

Verification of the second condition needs more steps. Let x_u be the position vector. When manipulating the second condition into a more suitable form we shall utilize the relation $x_{u,\ell} = \delta_{u\ell}$, the symmetry condition ${t^\star}_{p\ell} = {t^\star}_{\ell p}$ as well as the fulfillment of the equilibrium condition ${t^\star}_{p\ell,\ell} = 0$. For the moment $\mathcal{M}_v$ of the stresses about the origin we can now write by using the divergence theorem that

$$\mathcal{M}_v = \int_A x_u\epsilon_{upv}{t^\star}_{p\ell}n_\ell\,\mathrm{d}A = \int_V [\underbrace{x_{u,\ell}}_{\delta_{u\ell}}\epsilon_{upv}{t^\star}_{p\ell} + x_u\epsilon_{upv}\underbrace{{t^\star}_{p\ell,\ell}}_{=0}]\,\mathrm{d}V =$$

$$= -\int_V \underbrace{\epsilon_{p\ell v}{t^\star}_{p\ell}}_{=0}\,\mathrm{d}V = 0\,.$$

That was to be proved.

If the volume region V is bounded by more than one closed surfaces then the stresses are self-equilibrated on each closed surface. Consequently solution $\boldsymbol{t}^\star = -\nabla\times\boldsymbol{\mathcal{H}}\times\nabla$ can not be complete because there is no guarantee that the loads exerted on the separate closed surfaces are all self-equilibrated.

7.2.2. The general and complete solution. The general and complete solution to the equilibrium equations was published by H. Schaefer [43] in 1953. A mathematically different otherwise equivalent solution is given by M. Gurtin [56]. Papers [77, 78] by Szeidl and Kozák derive Schaefer's complete solution from the principle of complementary virtual work. Derivation of Schaefer's complete solution in this way is based partly on the knowledge of the necessary and sufficient conditions the strains should meet to be compatible. In this respect the line of thought in the papers cited has a methodological importance.

In what follows we shall consider Schaefer's solution.

Let $\boldsymbol{\chi}$ be a solution to the Poisson equation

$$\Delta\boldsymbol{\chi} = -\rho\mathbf{b}\,, \qquad \forall\mathbf{x} \in V\,. \tag{7.16}$$

It is well known that this problem always has a solution on V provided that the product $\rho\mathbf{b}$ meets the usual continuity conditions. Further let $\boldsymbol{\mathcal{H}}$ be, in the same way as earlier, an arbitrary symmetric tensor field on V. Then the stress field

$$\check{\boldsymbol{t}} = -\nabla\times\boldsymbol{\mathcal{H}}\times\nabla + \nabla\circ\boldsymbol{\chi} + \boldsymbol{\chi}\circ\nabla - (\boldsymbol{\chi}\cdot\nabla)\boldsymbol{1}\,, \qquad \forall\mathbf{x}\in V \tag{7.17}$$

is (a) equilibrated and (b) complete, that is, any solution of the equilibrium equations (7.14) can be given in the form (7.16).

First we shall show that the stress field $\check{\boldsymbol{t}}$ is equilibrated. Since the symmetric product $\boldsymbol{t}^{\star} = -\nabla\times\boldsymbol{\mathcal{H}}\times\nabla$ is a solution to the homogeneous equilibrium equations it is sufficient to show that

$$\boldsymbol{t}^{(p)} = \nabla\circ\boldsymbol{\chi} + \boldsymbol{\chi}\circ\nabla - (\boldsymbol{\chi}\cdot\nabla)\boldsymbol{1}\,, \qquad \forall\mathbf{x}\in V \tag{7.18}$$

is a particular solution. The symmetry is obvious. It also holds that

$$\boldsymbol{t}^{(p)}\cdot\nabla + \rho\mathbf{b} = (\boldsymbol{\chi}\cdot\nabla)\nabla + \underbrace{\boldsymbol{\chi}(\nabla\cdot\nabla)}_{-\rho\mathbf{b}} - (\boldsymbol{\chi}\cdot\nabla)\nabla + \rho\mathbf{b} = \mathbf{0}\,, \qquad \forall\mathbf{x}\in V \tag{7.19}$$

which means that (7.18) is really a particular solution.

In the sequel we shall consider the problem of completeness.

Let us assume that $\check{\boldsymbol{t}}$ is a solution of the equilibrium equations (7.14), i.e., it is an equilibrated, but not necessarily self-equilibrated stress field. Let $\boldsymbol{t}^{\times}$ be a solution of the Poisson equation

$$\Delta\boldsymbol{t}^{\times} = -\check{\boldsymbol{t}}\,, \qquad \forall\mathbf{x}\in V. \tag{7.20}$$

As it has already been mentioned in connection with equation (7.16) the above Poisson equation has a solution. It is also obvious that $\boldsymbol{t}^{\times}$ is a symmetric tensor. Given $\boldsymbol{t}^{\times}$, we define the stress function tensor $\boldsymbol{\mathcal{H}}$ by the following relation:

$$\boldsymbol{\mathcal{H}} = \boldsymbol{t}^{\times} - (\mathrm{tr}\,\boldsymbol{t}^{\times})\boldsymbol{1}\,, \qquad \forall\mathbf{x}\in V. \tag{7.21}$$

Because of the structural similarity between the expressions $-\nabla\times\boldsymbol{\varepsilon}\times\nabla$ and $-\nabla\times\boldsymbol{\mathcal{H}}\times\nabla$ we can apply relation (4.43) for manipulating the product $-\nabla\times\boldsymbol{\mathcal{H}}\times\nabla$ into a more suitable form:

$$\begin{aligned} -\nabla\times\boldsymbol{\mathcal{H}}\times\nabla = \left[(\mathrm{tr}\boldsymbol{\mathcal{H}})\nabla^2 - \nabla\cdot\boldsymbol{\mathcal{H}}\cdot\nabla\right]\boldsymbol{1} + \nabla\cdot\boldsymbol{\mathcal{H}}\circ\nabla + \nabla\circ\boldsymbol{\mathcal{H}}\cdot\nabla - \\ - \boldsymbol{\mathcal{H}}\nabla^2 - (\mathrm{tr}\,\boldsymbol{\mathcal{H}})\nabla\circ\nabla\,. \end{aligned}$$

Let us now substitute (7.21) for $\boldsymbol{\mathcal{H}}$ and take into account that

$$\boldsymbol{\mathcal{H}}\cdot\nabla = \boldsymbol{t}^{\times}\cdot\nabla - (\mathrm{tr}\,\boldsymbol{t}^{\times})\nabla\,, \qquad \mathrm{tr}\,\boldsymbol{\mathcal{H}} = -2\,\mathrm{tr}\,\boldsymbol{t}^{\times}\,. \tag{7.22}$$

We get

$$-\nabla\times\boldsymbol{\mathcal{H}}\times\nabla=\Big[\underline{-2\,(\mathrm{tr}\,\boldsymbol{t}^{\times})\Delta}-\nabla\cdot\boldsymbol{t}^{\times}\cdot\nabla+\underline{(\mathrm{tr}\,\boldsymbol{t}^{\times})\Delta}\Big]\,\boldsymbol{1}+$$
$$+\Big(\boldsymbol{t}^{\times}\cdot\nabla-\underbrace{(\mathrm{tr}\,\boldsymbol{t}^{\times})\nabla}\Big)\circ\nabla+\nabla\circ\Big(\boldsymbol{t}^{\times}\cdot\nabla-\underbrace{(\mathrm{tr}\,\boldsymbol{t}^{\times})\nabla}\Big)-$$
$$-\,\boldsymbol{t}^{\times}\Delta+\underline{(\mathrm{tr}\,\boldsymbol{t}^{\times})\Delta\boldsymbol{1}}+\underbrace{2\,(\mathrm{tr}\,\boldsymbol{t}^{\times})\nabla\circ\nabla},$$

where the terms underlined cancel out. Hence

$$-\nabla\times\boldsymbol{\mathcal{H}}\times\nabla=(\boldsymbol{t}^{\times}\cdot\nabla)\circ\nabla+\nabla\circ(\boldsymbol{t}^{\times}\cdot\nabla)-\nabla\cdot\boldsymbol{t}^{\times}\cdot\nabla-\underbrace{\Delta\boldsymbol{t}^{\times}}_{(7.20):\,-\check{\boldsymbol{t}}}$$

Let us introduce the notation $\boldsymbol{\chi}=-\boldsymbol{t}^{\times}\cdot\nabla$ and resolve the above equation for $\check{\boldsymbol{t}}$. We get:

$$\bar{\boldsymbol{t}}=-\nabla\times\boldsymbol{\mathcal{H}}\times\nabla+\nabla\circ\boldsymbol{\chi}+\boldsymbol{\chi}\circ\nabla-(\boldsymbol{\chi}\cdot\nabla)\boldsymbol{1}\,,\qquad\forall\mathbf{X}\in V.\tag{7.23}$$

Since this stress field satisfies equations (7.14) it also follows with regard to equation (7.22) that $\Delta\boldsymbol{\chi}=-\rho\mathbf{b}$. By this last remark the proof is full: we have shown that solution (7.17) is complete.

The body forces are said to be conservative if it holds that $\rho\mathbf{b}=-\varPsi\nabla$ where $\varPsi$ is the potential function. Making use of the identity $\varPsi\nabla=\varPsi\boldsymbol{1}\cdot\nabla$ we can rewrite equation (7.14) into the form

$$\boldsymbol{t}\cdot\nabla+\rho\mathbf{b}=(\boldsymbol{t}-\varPsi\boldsymbol{1})\cdot\nabla=\mathbf{0}\,.\tag{7.24}$$

It is obvious from equation (7.24) that the complete solution given by (7.23) belongs to the stress field $\boldsymbol{t}-\varPsi\boldsymbol{1}$. Consequently,

$$\bar{\boldsymbol{t}}=-\nabla\times\boldsymbol{\mathcal{H}}\times\nabla+\nabla\circ\boldsymbol{\chi}+\boldsymbol{\chi}\circ\nabla+(\varPsi-\boldsymbol{\chi}\cdot\nabla)\boldsymbol{1}\,,\qquad\forall\mathbf{X}\in V\tag{7.25}$$

is the general and complete solution of the equilibrium equations if the body forces are conservative.

The components of the stress function tensor $\boldsymbol{\mathcal{H}}$ are called stress functions.

Exercise 7.1: Derive formulae for the stress components in terms of the stress functions in the coordinate system (xyz).
It is not too difficult to verify using (7.25) that

$$\begin{aligned}
\sigma_{xx}&=\psi+\mathcal{H}_{yy,zz}+\mathcal{H}_{zz,yy}-2\mathcal{H}_{yz,yz}+\chi_{x,x}-\chi_{y,y}-\chi_{z,z}\,,\\
\sigma_{yy}&=\psi+\mathcal{H}_{zz,xx}+\mathcal{H}_{xx,zz}-2\mathcal{H}_{zx,zx}+\chi_{y,y}-\chi_{z,z}-\chi_{x,x}\,,\\
\sigma_{zz}&=\psi+\mathcal{H}_{xx,yy}+\mathcal{H}_{yy,xx}-2\mathcal{H}_{xy,xy}+\chi_{z,z}-\chi_{x,x}-\chi_{y,y}\,,\\
\tau_{xy}=\tau_{yx}&=\mathcal{H}_{yz,xz}+\mathcal{H}_{xz,yz}-\mathcal{H}_{xz,zz}-\mathcal{H}_{zz,xy}+\chi_{x,y}+\chi_{y,x}\,,\\
\tau_{yz}=\tau_{zy}&=\mathcal{H}_{zx,yx}+\mathcal{H}_{yx,zx}-\mathcal{H}_{yz,xx}-\mathcal{H}_{xx,yz}+\chi_{x,y}+\chi_{y,x}\,,\\
\tau_{zx}=\tau_{xz}&=\mathcal{H}_{xy,zy}+\mathcal{H}_{zy,xy}-\mathcal{H}_{zx,yy}-\mathcal{H}_{yy,xz}+\chi_{x,y}+\chi_{y,x}\,.
\end{aligned}\tag{7.26}$$

The general solution $\boldsymbol{t}^{\star}=-\nabla\times\boldsymbol{\mathcal{H}}\times\nabla$ of the homegenous equilibrium equations $\boldsymbol{t}\cdot\nabla=\mathbf{0}$ was found by A. Beltrami in 1892 [28].

Recalling that the kinematic equation (4.31) according to (4.46a) identically satisfies the compatibility equation and taking into account that the mathematical form of the compatibility equation $\boldsymbol{\eta} = -\nabla \times \boldsymbol{\varepsilon} \times \nabla = \mathbf{0}$ is the same as the solution $\boldsymbol{t}^\star = -\nabla \times \boldsymbol{\mathcal{H}} \times \nabla$ to the homogeneous equilibrium equations we can come to the conclusion that the same stress field is resulted from the stress function tensors

$$\boldsymbol{\mathcal{H}} \quad \text{and} \quad \boldsymbol{\mathcal{H}} + \frac{1}{2}(\mathbf{w} \circ \nabla + \nabla \circ \mathbf{w})$$

where $\mathbf{w}$ is an arbitrary differentiable vector field on V.

We can always select the vector field $\mathbf{w}$ in such a manner that three components of the stress function tensor are equal to zero. In a Cartesian coordinate system there are seven possibilities for selecting the non zero components of the stress function tensor.

Well known are the following two possibilities for selecting the non zero components of the stress function tensor:

– Maxwell's stress functions [19, 26]:

$$\underline{\boldsymbol{\mathcal{H}}} = \begin{bmatrix} \mathcal{H}_{11} & 0 & 0 \\ 0 & \mathcal{H}_{22} & 0 \\ 0 & 0 & \mathcal{H}_{33} \end{bmatrix}; \tag{7.27a}$$

Note that the off-diagonal elements are all set to zero.

– Morera's stress functions [30]:

$$\underline{\boldsymbol{\mathcal{H}}} = \begin{bmatrix} 0 & \mathcal{H}_{12} & \mathcal{H}_{13} \\ \mathcal{H}_{21} & 0 & \mathcal{H}_{23} \\ \mathcal{H}_{31} & \mathcal{H}_{12} & 0 \end{bmatrix} \tag{7.27b}$$

Here the diagonal elements are set to zero.

7.3. Principles of virtual power

7.3.1. Some formal manipulations. Assume that the stress tensor is dynamically admissible. Then

$$\bar{\boldsymbol{t}} \cdot \nabla + \mathbf{f}^{(a)} = \mathbf{0} \qquad \text{and} \qquad \bar{\boldsymbol{t}} = \bar{\boldsymbol{t}}^T, \tag{7.28a}$$

where

$$\mathbf{f}^{(a)} = \rho \mathbf{b} - \rho \mathbf{a}\,. \tag{7.28b}$$

Dot multiply equation $(7.28a)_1$ by the kinematically admissible velocity field $\hat{\mathbf{v}}$ and integrate the result over the volume V of the body. We get

$$\int_V \hat{\mathbf{v}} \cdot \left(\bar{\boldsymbol{t}} \cdot \nabla + \mathbf{f}^{(a)}\right) \mathrm{d}V = 0\,, \qquad \int_V \hat{v}_k \left(\bar{t}_{k\ell,\ell} + f_k^{(a)}\right) \mathrm{d}V = 0\,. \tag{7.29}$$

To obtain a point of departure for the principles of virtual power the above equation should be manipulated into a more suitable form. In accordance with the rule of partial integration (1.180) we can write

$$\int_V \hat{\mathbf{v}} \cdot (\overset{\downarrow}{\bar{\boldsymbol{t}}} \cdot \nabla)\, \mathrm{d}V = \int_V (\hat{\mathbf{v}} \cdot \bar{\boldsymbol{t}}) \cdot \nabla \mathrm{d}V - \int_V \overset{\downarrow}{\hat{\mathbf{v}}} \cdot \bar{\boldsymbol{t}} \cdot \nabla\, \mathrm{d}V \tag{7.30}$$

from where by applying divergence theorem (1.179) we have

$$\int_V \hat{\mathbf{v}} \cdot (\overset{\downarrow}{\bar{\boldsymbol{t}}} \cdot \nabla) \, \mathrm{d}V = \int_A \hat{\mathbf{v}} \cdot \bar{\boldsymbol{t}} \cdot \mathbf{n} \, \mathrm{d}A - \int_V \overset{\downarrow}{\hat{\mathbf{v}}} \cdot \bar{\boldsymbol{t}} \cdot \nabla \, \mathrm{d}V \, . \tag{7.31}$$

By utilizing equation $(1.96)_3$ we can rewrite the integrand of the last integral

$$\begin{aligned} \overset{\downarrow}{\hat{\mathbf{v}}} \cdot \bar{\boldsymbol{t}} \cdot \nabla = \bar{\boldsymbol{t}} \cdot\cdot (\hat{\mathbf{v}} \circ \nabla) = \bar{\boldsymbol{t}} \cdot\cdot \left[\frac{1}{2} (\hat{\mathbf{v}} \circ \nabla + \nabla \circ \hat{\mathbf{v}}) + \frac{1}{2} (\hat{\mathbf{v}} \circ \nabla - \nabla \circ \hat{\mathbf{v}}) \right] = \\ = \bar{\boldsymbol{t}} \cdot\cdot \left(\hat{\boldsymbol{d}} + \hat{\boldsymbol{\Omega}} \right) = \bar{\boldsymbol{t}} \cdot\cdot \hat{\boldsymbol{d}} + \underbrace{\bar{\boldsymbol{t}} \cdot\cdot \hat{\boldsymbol{\Omega}}}_{=0} = \bar{\boldsymbol{t}} \cdot\cdot \hat{\boldsymbol{d}} \, . \end{aligned} \tag{7.32}$$

Hence,

$$\begin{aligned} \int_V \hat{\mathbf{v}} \cdot (\overset{\downarrow}{\bar{\boldsymbol{t}}} \cdot \nabla) \, \mathrm{d}V &= \int_A \hat{\mathbf{v}} \cdot \bar{\boldsymbol{t}} \cdot \mathbf{n} \, \mathrm{d}A - \int_V \bar{\boldsymbol{t}} \cdot\cdot \hat{\boldsymbol{d}} \, \mathrm{d}V \, , \\ \int_V \hat{v}_k \, \bar{t}_{k\ell,\ell} \, \mathrm{d}V &= \int_A \hat{v}_k \, \bar{t}_{k\ell} \, n_\ell \, \mathrm{d}A - \int_V \bar{t}_{k\ell} \hat{d}_{k\ell} \, \mathrm{d}V \, . \end{aligned} \tag{7.33}$$

If we substitute this result back into (7.29) after a rearrangement we obtain

$$\begin{aligned} \int_V \bar{\boldsymbol{t}} \cdot\cdot \hat{\boldsymbol{d}} \, \mathrm{d}V &= \int_V \hat{\mathbf{v}} \cdot \mathbf{f}^{(a)} \, \mathrm{d}V + \int_A \hat{\mathbf{v}} \cdot \bar{\boldsymbol{t}} \cdot \mathbf{n} \, \mathrm{d}A \, , \\ \int_V \bar{t}_{k\ell} \hat{d}_{k\ell} \, \mathrm{d}V &= \int_V \hat{v}_k f_k^{(a)} \, \mathrm{d}V + \int_A \hat{v}_k \bar{t}_{k\ell} n_\ell \, \mathrm{d}A \, . \end{aligned} \tag{7.34}$$

REMARK 7.5: Since we have not utilized the boundary conditions so far the kinematically admissible velocity field $\hat{\mathbf{v}}$ and the dynamically admissible stress field $\bar{\boldsymbol{t}}$ may belong to different boundary value problems (the resolution of the boundary surface A into two parts A_v and A_t can also be different for the two boundary value problems mentioned).

Let us assume that $\hat{\mathbf{v}}$ and $\bar{\boldsymbol{t}}$ belong to the same boundary value problem. Then $\hat{\mathbf{v}} = \tilde{\mathbf{v}}$ on A_v and $\bar{\boldsymbol{t}} \cdot \mathbf{n} = \tilde{\mathbf{t}}$ on A_t. Because of the additivity of the surface integral we can rewrite equation (7.34) into the form

$$\boxed{\begin{aligned} \int_V \bar{\boldsymbol{t}} \cdot\cdot \hat{\boldsymbol{d}} \, \mathrm{d}V &= \int_V \hat{\mathbf{v}} \cdot \mathbf{f}^{(a)} \, \mathrm{d}V + \int_{A_v} \tilde{\mathbf{v}} \cdot \bar{\boldsymbol{t}} \cdot \mathbf{n} \, \mathrm{d}A + \int_{A_t} \hat{\mathbf{v}} \cdot \tilde{\mathbf{t}} \, \mathrm{d}A \, , \\ \int_V \bar{t}_{k\ell} \hat{d}_{k\ell} \, \mathrm{d}V &= \int_V \hat{v}_k f_k^{(a)} \, \mathrm{d}V + \int_{A_v} \tilde{v}_k \bar{t}_{k\ell} n_\ell \, \mathrm{d}A + \int_{A_t} \hat{v}_k \tilde{t}_k \, \mathrm{d}A \, . \end{aligned}} \tag{7.35}$$

This equation holds for any kinematically admissible velocity field $\hat{\mathbf{v}}$, strain rate field $\hat{\boldsymbol{d}}$ as well as for any dynamically admissible (statically admissible if $\mathbf{a} = \mathbf{0}$) stress field $\bar{\boldsymbol{t}}$.

It is worthy of mention that the above equation is independent of the constitutive equations.

The left sides of equations (7.34), (7.35) are the fictitious stress powers that the dynamically admissible stress field $\bar{\boldsymbol{t}}$ has on a kinematically admissible strain rate field $\hat{\boldsymbol{d}}$ (the adjective fictitious means that none of the two fields mentioned are, in general, true solutions) whereas the right sides are the fictitious power of the external forces on a kinematically admissible velocity field $\hat{\mathbf{v}}$.

7.3.2. Principle of virtual power. If we have two kinematically admissible velocity fields we may write:

$$\int_V \bar{\boldsymbol{t}}\cdot\cdot\,\hat{\boldsymbol{d}}_{II}\,\mathrm{d}V = \int_V \mathbf{f}^{(a)}\cdot\hat{\mathbf{v}}_{II}\,\mathrm{d}V + \int_{A_v}\tilde{\mathbf{v}}\cdot\bar{\boldsymbol{t}}\cdot\mathbf{n}\,\mathrm{d}A + \int_{A_t}\hat{\mathbf{v}}_{II}\cdot\tilde{\mathbf{t}}\,\mathrm{d}A\,, \tag{7.36a}$$

$$\int_V \bar{\boldsymbol{t}}\cdot\cdot\,\hat{\boldsymbol{d}}_{I}\,\mathrm{d}V = \int_V \mathbf{f}^{(a)}\cdot\hat{\mathbf{v}}_{I}\,\mathrm{d}V + \int_{A_v}\tilde{\mathbf{v}}\cdot\bar{\boldsymbol{t}}\cdot\mathbf{n}\,\mathrm{d}A + \int_{A_t}\hat{\mathbf{v}}_{I}\cdot\tilde{\mathbf{t}}\,\mathrm{d}A\,. \tag{7.36b}$$

Subtract now the second equation from the first one by taking the relations

$$\hat{\mathbf{v}}_{II}-\hat{\mathbf{v}}_I=\delta\mathbf{v}\,,\qquad \hat{\boldsymbol{d}}_{II}-\hat{\boldsymbol{d}}_I=\delta\boldsymbol{d}$$

into account. Since $\delta\mathbf{v}=\mathbf{0}$ on A_v we get

$$\int_V \bar{\boldsymbol{t}}\cdot\cdot\,\delta\boldsymbol{d}\,\mathrm{d}V = \int_V \delta\mathbf{v}\cdot\mathbf{f}^{(a)}\,\rho\mathrm{d}V + \int_{A_t}\delta\mathbf{v}\cdot\tilde{\mathbf{t}}\,\mathrm{d}A\,. \tag{7.37}$$

Assume that given $\mathbf{f}^{(a)}$ $(\mathbf{x}\in V)$ and $\tilde{\boldsymbol{t}}$ $(\mathbf{x}\in A_t)$ $(A_v\cup A_t=A,\ A_v\cap A_t=0)$ in the considered configuration of the continuum. Then we may state that equation (7.37) holds for any dynamically admissible stress field $\bar{\boldsymbol{t}}$ and virtual velocity field $\delta\mathbf{v}$.

Assume further (a) that given the vector field $\mathbf{f}^{(a)}$ $(\mathbf{x}\in V)$, (b) that the stress field $\boldsymbol{t}(x)$, which we regard a fixed quantity, fulfills the symmetry condition $\boldsymbol{t}=\boldsymbol{t}^T$ $\mathbf{x}\in V$ and finally (c) that the equation

$$\boxed{\begin{aligned}&\int_V \boldsymbol{t}\cdot\cdot\,\delta\boldsymbol{d}\,\mathrm{d}V = \int_V \mathbf{f}^{(a)}\cdot\delta\mathbf{v}\,\rho\mathrm{d}V + \int_{A_t}\tilde{\mathbf{t}}\cdot\delta\mathbf{v}\,\mathrm{d}A\,,\\ &\int_V t_{k\ell}\,\delta d_{k\ell}\,\mathrm{d}V = \int_V f_k^{(a)}\delta v_k\,\rho\mathrm{d}V + \int_{A_t}\tilde{t}_k\delta v_k\,\mathrm{d}A\end{aligned}} \tag{7.38a}$$

holds for any $\delta\mathbf{v}$ under the side conditions

$$\begin{aligned}\delta\boldsymbol{d}&=\frac{1}{2}\left(\delta\mathbf{v}\circ\nabla+\nabla\circ\delta\mathbf{v}\right)\,, &&\mathbf{x}\in V\,; &&\delta\mathbf{v}=\mathbf{0}\,, &&\mathbf{x}\in A_v\,,\\ \delta d_{k\ell}&=\frac{1}{2}\left(\delta v_{k,\ell}+\delta v_{\ell,k}\right)\,, &&\mathbf{x}\in V\,; &&\delta v_k=0\,, &&\mathbf{x}\in A_v\,.\end{aligned} \tag{7.38b}$$

Then the stress field $\boldsymbol{t}$ is dynamically admissible.

To prove this statement substitute the manipulation

$$\int_V t_{k\ell}\delta d_{k\ell}\,\mathrm{d}V = \int_V t_{k\ell}\frac{1}{2}\left(\delta v_{k,\ell}+\delta v_{\ell,k}\right)\,\mathrm{d}V + \underbrace{\int_V t_{k\ell}\frac{1}{2}\left(\delta v_{k,\ell}-\delta v_{\ell,k}\right)\,\mathrm{d}V}_{t_{k\ell}\delta\Omega_{k\ell}=0} =$$

$$= \int_V t_{k\ell}\delta v_{k,\ell}\,\mathrm{d}V = \int_V \left(t_{k\ell}\,\delta v_k\right)_{,\ell}\,\mathrm{d}V - \int_V t_{k\ell,\ell}\,\delta v_k\,\mathrm{d}V = \underset{(1.179)}{\uparrow} =$$

$$= \int_{A=A_t\cup A_v} t_{k\ell}n_\ell\,\delta v_k\,\mathrm{d}A - \int_V t_{k\ell,\ell}\,\delta v_k\,\mathrm{d}V = \underset{(7.10a)}{\uparrow} =$$

$$= \int_{A_t} t_{k\ell} n_\ell \, \delta v_k \, \mathrm{d}A - \int_V t_{k\ell,\ell} \, \delta v_k \, \mathrm{d}V \qquad (7.39)$$

into (7.38b). After a rearrangement we obtain

$$\int_V \delta v_k \left(t_{k\ell,\ell} + f_k^{(a)} \right) \mathrm{d}V + \int_{A_t} \delta v_k \left(\tilde{t}_k - t_{k\ell} n_\ell \right) \mathrm{d}A = 0 \,. \qquad (7.40)$$

Here the two integrals are taken over different regions and δv_k is arbitrary. Consequently,

$$t_{k\ell,\ell} + f_k^{(a)} = 0, \qquad x_p \in V \qquad \text{and} \qquad \tilde{t}_k = t_{k\ell} n_\ell \qquad x_p \in A_t \qquad (7.41)$$

which means that $t_{k\ell}$ is dynamically admissible. That was to be proved.

The previous statement (the statement we have proved) is the principle of virtual power. It is worthy of mentioning that this form of the principle is also independent of the constitutive equations.

7.3.3. Principle of complementary virtual power. If we have two dynamically admissible stress fields equation (7.34) yields

$$\int_V \bar{\boldsymbol{t}}_{II} \cdot\cdot \hat{\boldsymbol{d}} \, \mathrm{d}V = \int_V \mathbf{f}^{(a)} \cdot \hat{\mathbf{v}} \, \mathrm{d}V + \int_{A_v} \tilde{\mathbf{v}} \cdot \bar{\boldsymbol{t}}_{II} \cdot \mathbf{n} \, \mathrm{d}A + \int_{A_t} \hat{\mathbf{v}} \cdot \tilde{\mathbf{t}} \, \mathrm{d}A \,, \qquad (7.42a)$$

$$\int_V \bar{\boldsymbol{t}}_{I} \cdot\cdot \hat{\boldsymbol{d}} \, \mathrm{d}V = \int_V \mathbf{f}^{(a)} \cdot \hat{\mathbf{v}} \, \mathrm{d}V + \int_{A_v} \tilde{\mathbf{v}} \cdot \bar{\boldsymbol{t}}_{I} \cdot \mathbf{n} \, \mathrm{d}A + \int_{A_t} \hat{\mathbf{v}} \cdot \tilde{\mathbf{t}} \, \mathrm{d}A \,. \qquad (7.42b)$$

It is worthy of subtracting the second equation from the first one by utilizing the relation

$$\bar{\boldsymbol{t}}_{II} - \bar{\boldsymbol{t}}_{I} = \delta \boldsymbol{t}$$

and taking into account that the virtual stress field $\delta \boldsymbol{t}$ is zero on A_t. We obtain

$$\int_V \hat{\boldsymbol{d}} \cdot\cdot \delta \boldsymbol{t} \, \mathrm{d}V = \int_{A_v} \tilde{\mathbf{v}} \cdot \delta \boldsymbol{t} \cdot \mathbf{n} \, \mathrm{d}A \,. \qquad (7.43)$$

Assume that given $\tilde{\boldsymbol{v}}$ $(\mathbf{x} \in A_v)$ $(A_v \cup A_t = A,\ A_v \cap A_t = 0)$ in the considered configuration of the continuum. Then we may state that equation (7.43) holds for any kinematically admissible velocity field $\hat{\mathbf{v}}$ and virtual stress field $\delta \boldsymbol{t}$.

Assume further (a) that given $\tilde{\mathbf{v}}$ $(\mathbf{x} \in A_v)$, (b) that the strain rate tensor, which we regard a fixed quantity, fulfills the symmetry condition $\boldsymbol{d} = \boldsymbol{d}^T$ $(\mathbf{x} \in V)$ and finally (c) that equation

$$\boxed{\begin{aligned} \int_V \boldsymbol{d} \cdot\cdot \delta \boldsymbol{t} \, \mathrm{d}V &= \int_{A_v} \tilde{\mathbf{v}} \cdot \delta \boldsymbol{t} \cdot \mathbf{n} \, \mathrm{d}A \,, \\ \int_V d_{k\ell} \delta t_{k\ell} \, \mathrm{d}V &= \int_{A_v} \tilde{v}_k \delta t_{k\ell} n_\ell \, \mathrm{d}A \,, \end{aligned}} \qquad (7.44a)$$

holds for any $\delta\boldsymbol{t}$ under the side conditions

$$\begin{aligned} &\delta\boldsymbol{t}\cdot\nabla=\mathbf{0}\,, \qquad \delta\boldsymbol{t}=\delta\boldsymbol{t}^T\,, && \mathbf{x}\in V,\\ &\qquad\quad \delta\boldsymbol{t}\cdot\mathbf{n}=\mathbf{0}\,, && \mathbf{x}\in A_t\,.\\ &\delta t_{k\ell,\ell}=0\,, \qquad \delta t_{k\ell}=\delta t_{\ell,k}\,, && x_q\in V\,,\\ &\qquad\quad \delta t_{k\ell}n_\ell=0\,, && x_q\in A_t\,. \end{aligned} \tag{7.44b}$$

Then the strain rate tensor is kinematically admissible. This statement is the principle of complementary virtual power.

Remark 7.6: We shall not prove this statement.

Remark 7.7: For $\mathbf{a}=\mathbf{0}$ $\mathbf{f}^{(a)}=\mathbf{f}=\rho\mathbf{b}$. The previous equations and statements remain valid for this case as well provided that the expression dynamically admissible is changed to the expression statically admissible.

Remark 7.8: It is worthy of drawing the attention to the following fact. Whereas (7.35), (7.37) and (7.43) are fulfilled by only one-one $\bar{\boldsymbol{t}}$, $\hat{\mathbf{v}}$; or $\bar{\boldsymbol{t}}$, $\delta\mathbf{v}$; or $\hat{\boldsymbol{d}}$, $\delta\boldsymbol{v}$ tensor and vector fields the statements of the virtual power and complementary virtual power require the examination of the all (that is infinitely many) virtual velocity fields $\delta\mathbf{v}$ and stress fields $\delta\boldsymbol{t}$. In spite of this the latter two principles have significant role when we seek approximate solutions and instead of examining the totality of the possible vector and tensor fields we select only a few which are, however, best characterize the problem to be investigated. We mention that the same is valid for the virtual work principles presented in Section 7.4.4.

7.3.4. Equations in the initial configuration. The equations and statements we have presented in Sections 7.3.1, 7.3.4 and 7.3.3 are related to the current configuration of the body. On the basis of equations (6.44), (2.94) and (2.92), however, we can transform the principle of virtual power (7.38a) into the initial configuration:

$$\boxed{\int_{V^\circ}\boldsymbol{S}\cdot\cdot\,(\delta\boldsymbol{E})^{\boldsymbol{\cdot}}\,\mathrm{d}V^\circ=\int_{V^\circ}\lambda_V\,\mathbf{f}^{(a)}\cdot\delta\mathbf{v}^\circ\,\mathrm{d}V^\circ+\int_{A_t^\circ}\lambda_A\,\tilde{\mathbf{t}}\cdot\delta\mathbf{v}^\circ\,\mathrm{d}A^\circ\,.} \tag{7.45}$$

Making use of equations (5.22), (5.23) and (5.23) we may also write:

$$\begin{aligned}\mathrm{d}\mathbf{F}^S=\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A=J\,\boldsymbol{t}\cdot\boldsymbol{F}^{-T}\cdot\mathbf{n}^\circ\mathrm{d}A^\circ=J\,\underbrace{\boldsymbol{F}\cdot\boldsymbol{F}^{-1}}_{\boldsymbol{1}}\boldsymbol{t}\cdot\boldsymbol{F}^{-T}\cdot\mathbf{n}^\circ\mathrm{d}A^\circ=\\ =J\,\boldsymbol{F}\cdot\underbrace{J\boldsymbol{F}^{-1}\boldsymbol{t}\cdot\boldsymbol{F}^{-T}}_{\boldsymbol{S}}\cdot\mathbf{n}^\circ\mathrm{d}A^\circ=\boldsymbol{F}\cdot\boldsymbol{S}\cdot\mathbf{n}^\circ\mathrm{d}A^\circ\,.\end{aligned} \tag{7.46}$$

Hence

$$\delta\boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A=\boldsymbol{F}\cdot\delta\boldsymbol{S}\cdot\mathbf{n}^\circ\mathrm{d}A^\circ\,. \tag{7.47}$$

With this equation we can also give the principle of complementary virtual power (7.44) in the initial configuration:

$$\boxed{\int_{V^\circ}(\boldsymbol{E})^{\boldsymbol{\cdot}}\cdot\cdot\,\delta\boldsymbol{S}\,\mathrm{d}V^\circ=\int_{A_v^\circ}\tilde{\mathbf{v}}\cdot\boldsymbol{F}\cdot\delta\boldsymbol{S}\cdot\mathbf{n}^\circ\mathrm{d}A^\circ\,.} \tag{7.48}$$

It is necessary to emphasize that equations (7.45) and (7.48) concern the considered configuration (the initial configuration) of the body, i.e., they are valid for given λ_V, λ_A and $\boldsymbol{F}$. When applying equations (7.45) and (7.48) the quantities defined in the current configuration – for instance $\delta\mathbf{v}$, $\mathbf{f}^{(a)}$ – should all, therefore, be transformed into the initial configuration.

7.4. Virtual work principles

7.4.1. Principle of virtual work in the current configuration. Let δt be a very small time amount. Then

$$\delta\mathbf{u} = \delta\mathbf{v}\,\delta t\,. \tag{7.49}$$

Multiplying equation (7.38a) by δt and taking into account that

$$\delta t\,\delta\boldsymbol{d} = \frac{1}{2}\left(\delta\mathbf{u}\circ\nabla + \nabla\circ\delta\mathbf{u}\right) = \delta\boldsymbol{e}^L \qquad \mathbf{x}\in V \tag{7.50}$$

we get

$$\int_V \boldsymbol{t}\cdot\cdot\,\delta\boldsymbol{e}^L\,\mathrm{d}V = \int_V \mathbf{f}^{(a)}\cdot\delta\mathbf{u}\,\rho\mathrm{d}V + \int_{A_t}\tilde{\mathbf{t}}\cdot\delta\mathbf{u}\,\mathrm{d}A \tag{7.51}$$

in which $\delta\boldsymbol{e}^L$ is the virtual change for the linear part of the Euler-Almansi strain tensor.

Assume that given the vector field $\mathbf{f}^{(a)}$ ($\mathbf{x}\in V$). Assume further that the stress field $\boldsymbol{t}(x)$, which we regard a fixed quantity, fulfills the symmetry condition $\boldsymbol{t} = \boldsymbol{t}^T \quad \mathbf{x}\in V$. If equation (7.51) holds for any $\delta\mathbf{u}$ under the side conditions

$$\delta\boldsymbol{e}^L = \frac{1}{2}\left(\delta\mathbf{u}\circ\nabla + \nabla\circ\delta\mathbf{u}\right)\,,\quad \mathbf{x}\in V\,;\qquad \delta\mathbf{u} = \mathbf{0}\,,\quad \mathbf{x}\in A_v \tag{7.52}$$

then the stress field is dynamically admissible (or statically admissible if $\mathbf{a} = \mathbf{0}$).

The prof of this statement is basically the same as that of the principle of virtual power. For this reason it is omitted.

Remark 7.9: We shall prove that $\delta\boldsymbol{e}^L$ coincides with the variation of the Euler-Almansi strain tensor. On the basis of equation $(2.68)_2$ we have

$$\delta e_{k\ell} = F^{-1}_{kA}\,\delta E_{AB}\,F^{-1}_{B\ell} \tag{7.53}$$

where with regard to $(2.32)_{3,4}$ it holds that

$$\begin{aligned}\delta E_{AB} &= \delta\frac{1}{2}\left(F_{Ar}F_{rB} + \delta_{AB}\right) = \frac{1}{2}\left(\delta F_{Ar}\right)F_{rB} + \frac{1}{2}F_{Ar}\left(\delta F_{rB}\right) = \\ &\underset{\boldsymbol{F}=\boldsymbol{I}+\mathbf{u}^\circ\circ\nabla^\circ}{=\uparrow=} \frac{1}{2}\left(\nabla^\circ_A\delta u^\circ_R\right)F_{rB} + \frac{1}{2}F_{Ar}\left(\delta u^\circ_R\nabla^\circ_B\right)\,.\end{aligned}$$

Substituting the above relationship into equation (7.53) by taking into account that $\delta\mathbf{u}^\circ = \delta\boldsymbol{u}$ and $R = r$ we get

$$\delta e_{k\ell} = \frac{1}{2} F^{-1}_{kA} \left(\nabla^\circ_A \delta u_r\right) \underbrace{F_{rB} F^{-1}_{B\ell}}_{\delta_{r\ell}} + \frac{1}{2} \underbrace{F^{-1}_{kA} F_{Ar}}_{\delta_{kr}} \left(\delta u_r \nabla^\circ_B\right) F^{-1}_{B\ell} =$$

$$= \frac{1}{2}\left[F^{-1}_{kA} \left(\nabla^\circ_A \delta u_\ell\right) + \left(\delta u_k \nabla^\circ_A\right) F^{-1}_{A\ell}\right]. \quad (7.54)$$

According to equation (2.30) $\nabla^\circ_A F^{-1}_{A\ell} = \nabla_\ell$ while $F^{-1}_{kA}\left(\nabla^\circ_A \delta u_\ell\right)$ is the transpose of $\left(\delta u_k \nabla^\circ_A\right) F^{-1}_{A\ell}$. Hence, we have

$$\delta e_{k\ell} = \frac{1}{2}\left[\delta u_k \nabla_\ell + \nabla_k \delta u_\ell\right]. \quad (7.55)$$

7.4.2. Principle of virtual work in the initial configuration. With regard to (7.28b), (6.14) and (6.15) it holds that

$$\mathbf{f}^{(a)} \cdot \delta\mathbf{v}\, \mathrm{d}V = (\mathbf{b} - \mathbf{a}) \cdot \delta\mathbf{v}\, \rho \mathrm{d}V = (\mathbf{b}^\circ - \mathbf{a}^\circ) \cdot \delta\mathbf{v}^\circ\, \rho^\circ \mathrm{d}V^\circ \quad (7.56)$$

where $\mathbf{b} = \mathbf{b}^\circ$, $\mathbf{a} = \mathbf{a}^\circ$ and $\delta\mathbf{v} = \delta\mathbf{v}^\circ$. The surface tractions $\tilde{\mathbf{t}}^\circ(\mathbf{X})$ in the initial configuration are defined by the following equation:

$$\tilde{\mathbf{t}}^\circ(\mathbf{X}) = \lambda_A\, \tilde{\mathbf{t}}\,. \quad (7.57)$$

Substituting (7.56) and (7.57) into (7.45) yields

$$\int_{V^\circ} \boldsymbol{S} \cdot\cdot\, (\delta\boldsymbol{E})^\cdot\, \mathrm{d}V^\circ = \int_{V^\circ} (\mathbf{b}^\circ - \mathbf{a}^\circ) \cdot \delta\mathbf{v}^\circ\, \rho^\circ \mathrm{d}V^\circ + \int_{A^\circ_t} \tilde{\mathbf{t}}^\circ \cdot \delta\mathbf{v}^\circ\, \mathrm{d}A^\circ. \quad (7.58)$$

Let δt be a very small time amount. Then

$$\delta\mathbf{u}^\circ = \delta\mathbf{u} = \delta\mathbf{v}\, \delta t\,, \qquad \delta\boldsymbol{E} = (\delta\boldsymbol{E})^\cdot\, \delta t\,. \quad (7.59)$$

Multiply equation (7.58) by δt. We get

$$\boxed{\begin{aligned} \int_{V^\circ} \boldsymbol{S} \cdot\cdot\, \delta\boldsymbol{E}\, \mathrm{d}V^\circ &= \int_{V^\circ} (\mathbf{b}^\circ - \mathbf{a}^\circ) \cdot \delta\mathbf{u}^\circ\, \rho^\circ \mathrm{d}V^\circ + \int_{A^\circ_t} \tilde{\mathbf{t}}^\circ \cdot \delta\mathbf{u}^\circ\, \mathrm{d}A^\circ, \\ \int_{V^\circ} S_{AB}\, \delta E_{AB}\, \mathrm{d}V^\circ &= \int_{V^\circ} (b^\circ_M - a^\circ_M)\, \delta u^\circ_M\, \rho^\circ \mathrm{d}V^\circ + \int_{A^\circ_t} \tilde{t}^\circ_M \delta u^\circ_M\, \mathrm{d}A^\circ. \end{aligned}} \quad (7.60)$$

Assume that (a) given the vector field $\rho^\circ(\mathbf{b}^\circ - \mathbf{a}^\circ)$ $(\mathbf{X} \in V^\circ)$ and (b) the stress field $\boldsymbol{S}(\mathbf{X})$, which is regarded as a fixed quantity and fulfills the symmetry condition $\boldsymbol{S} = \boldsymbol{S}^T$ $(\mathbf{X} \in V^\circ)$. Assume further that the equation (7.60) holds for any $\delta\mathbf{u}^\circ$ under the side conditions

$$\delta\boldsymbol{E} = \frac{1}{2}\big[\delta\mathbf{u}^\circ \circ \nabla^\circ + \nabla^\circ \circ \delta\mathbf{u}^\circ +$$

$$+ (\nabla^\circ \circ \delta\mathbf{u}^\circ) \cdot (\mathbf{u}^\circ \circ \nabla^\circ) + (\nabla^\circ \circ \mathbf{u}^\circ) \cdot (\delta\mathbf{u}^\circ \circ \nabla^\circ)\big], \quad \mathbf{X} \in V^\circ, \quad (7.61a)$$

$$\delta\mathbf{u}^\circ = \mathbf{0}\,, \qquad \mathbf{X} \in A^\circ_u\,, \quad (7.61b)$$

where the virtual Green-Lagrange strain tensor $\delta\boldsymbol{E}$ is taken from equation (7.12). Then the stress field $\boldsymbol{S}$ is dynamically admissible (statically if $\mathbf{a} = \mathbf{0}$).

This statement is the principle of virtual work in the initial configuration.

Substitute side condition (7.61a) into the volume integral on the left side of equation (7.60) and take the symmetry of the second Piola-Kirchhoff stress tensor S_{AB} into account. If in addition to this we take (2.24) also into account we get

$$\begin{aligned}\int_{V^\circ} S_{AB}\,\delta E_{AB}\,\mathrm{d}V^\circ &= \\ &= \int_{V^\circ} S_{AB}\,\frac{1}{2}\left(\delta u^\circ_{A,B} + \delta u^\circ_{B,A} + \delta u^\circ_{M,A}\,u^\circ_{M,B} + u^\circ_{M,A}\,\delta u^\circ_{M,B}\right)\mathrm{d}V^\circ = \\ &= \int_{V^\circ} S_{AB}\,\left(\delta u^\circ_{A,B} + \delta u^\circ_{M,B}\,u^\circ_{M,A}\right)\mathrm{d}V^\circ = \\ &= \int_{V^\circ} \left(\delta_{MA}\delta u^\circ_{M,B} + \delta u^\circ_{M,B}\,u^\circ_{M,A}\right)S_{AB}\,\mathrm{d}V^\circ = \\ &= \int_{V^\circ} \left(\delta_{MA} + u^\circ_{M,A}\right)\,S_{AB}\,\delta u^\circ_{M,B}\mathrm{d}V^\circ = \int_{V^\circ} F_{MA}\,S_{AB}\,\delta u^\circ_{M,B}\mathrm{d}V^\circ\end{aligned}$$

in which F_{MA} is the deformation gradient in the initial configuration. By performing partial integration it follows from here that

$$\begin{aligned}\int_{V^\circ} S_{AB}\,\delta E_{AB}\,\mathrm{d}V^\circ &= \\ &= \int_{V^\circ} \left(F_{MA}\,S_{AB}\,\delta u^\circ_M\right)_{,B}\mathrm{d}V^\circ - \int_{V^\circ}\left(F_{MA}\,S_{AB}\right)_{,B}\,\delta u^\circ_M\mathrm{d}V^\circ = \\ &= \int_{A^\circ = A^\circ_u \cup A^\circ_t} F_{MA}\,S_{AB}\,n^\circ_B\,\delta u^\circ_M\,\mathrm{d}A^\circ - \int_{V^\circ}\left(F_{MA}\,S_{AB}\right)_{,B}\,\delta u^\circ_M\,\mathrm{d}V^\circ = \\ &= \int_{A^\circ_t} F_{MA}\,S_{AB}\,n^\circ_B\,\delta u^\circ_M\,\mathrm{d}A^\circ - \int_{V^\circ}\left(F_{MA}\,S_{AB}\right)_{,B}\,\delta u^\circ_M\,\mathrm{d}V^\circ \end{aligned} \tag{7.62}$$

since $\delta u^\circ_M = 0$ on A°_u. Substitution of (7.62) into (7.60) leads to the following result

$$\begin{aligned}\int_{V^\circ}\left[\left(F_{MA}\,S_{AB}\right)_{,B} + \rho^\circ\left(b^\circ_M - a^\circ_M\right)\right]\,\delta u^\circ_M\,\mathrm{d}V^\circ + \\ + \int_{A^\circ_t}\left[\tilde{t}^\circ_M - F_{MA}\,S_{AB}\,n^\circ_B\right]\,\delta u^\circ_M\,\mathrm{d}A^\circ = 0\,,\end{aligned} \tag{7.63}$$

where δu°_M is arbitrary. Hence

$$\left(F_{MA}\,S_{AB}\right)_{,B} + \rho^\circ b^\circ_M = \rho^\circ a^\circ_M\,, \qquad X_L \in V^\circ \tag{7.64a}$$

and

$$\tilde{t}^\circ_M = F_{MA}\,S_{AB}\,n^\circ_B\,, \qquad X_L \in A^\circ_t\,. \tag{7.64b}$$

Equation [(7.64a)] {(7.64b)} is the [equation of motion] {the traction boundary condition} in the initial configuration. If they are fulfilled the stress field S_{AB} is dynamically admissible (statically if $a^\circ_M = 0$). This was to be proven.

REMARK 7.10: Within the framework of the linear theory $\boldsymbol{S} = \boldsymbol{\sigma}$, $\delta\boldsymbol{E} = \delta\boldsymbol{\varepsilon}$, $\lambda_V = \lambda_A = 1$, $\nabla^\circ = \nabla$, $V^\circ = V$, $A_t^\circ = A_t$, $A_u^\circ = A_u$ and $\rho = \rho^\circ$. With these in mind we can rewrite the virtual work principle (7.60) into the following form:

$$\int_V \boldsymbol{\sigma} \cdot\cdot\, \delta\boldsymbol{\varepsilon}\, \mathrm{d}V = \int_V \mathbf{f}^{(a)} \cdot \delta\mathbf{u}\, \mathrm{d}V + \int_{A_t} \tilde{\mathbf{t}} \cdot \delta\mathbf{u}\, \mathrm{d}A\,. \tag{7.65}$$

Assume that (a) given the vector field $\mathbf{f}^{(a)}$ $(\mathbf{x} \in V)$ and (b) the stress field $\boldsymbol{\sigma}(x)$, which is regarded as a fixed quantity and fulfills the symmetry condition $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \;\; \mathbf{x} \in V$. Assume further that equation (7.65) holds for any $\delta\mathbf{u}$ under the side conditions

$$\delta\boldsymbol{\varepsilon} = \frac{1}{2}\left(\delta\mathbf{u} \circ \nabla + \nabla \circ \delta\mathbf{u}\right)\,, \quad \mathbf{x} \in V\,; \qquad \delta\mathbf{u} = \mathbf{0}\,, \quad \mathbf{x} \in A_u\,. \tag{7.66}$$

Then the stress field $\boldsymbol{\sigma}$ is dynamically admissible (statically if $\mathbf{a} = \mathbf{0}$, i.e., $\mathbf{f}^{(a)} = \mathbf{f} = \rho\mathbf{b}$).

7.4.3. Principle of complementary virtual work. We present the principle of the complementary virtual work within the framework of the linear theory only. On the basis of equations (7.44) we may write

$$\begin{aligned} \int_V \boldsymbol{\varepsilon} \cdot\cdot\, \delta\boldsymbol{\sigma}\, \mathrm{d}V &= \int_{A_u} \tilde{\mathbf{u}} \cdot \delta\boldsymbol{\sigma} \cdot \mathbf{n}\, \mathrm{d}A\,, \\ \int_V \varepsilon_{k\ell}\, \delta\sigma_{k\ell}\, \mathrm{d}V &= \int_{A_u} \tilde{u}_k\, \delta\sigma_{k\ell}\, n_\ell\, \mathrm{d}A\,, \end{aligned} \tag{7.67a}$$

where

$$\begin{aligned} &\delta\boldsymbol{\sigma} \cdot \nabla = \mathbf{0}\,, && \delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma}^T\,, && \mathbf{x} \in V, \\ &&& \delta\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}\,, && \mathbf{x} \in A_t\,. \\ &\delta\sigma_{k\ell,\ell} = 0\,, && \delta\sigma_{k\ell} = \delta\sigma_{\ell,k}\,, && x_q \in V, \\ &&& \delta\sigma_{k\ell} n_\ell = 0\,, && x_q \in A_t\,. \end{aligned} \tag{7.67b}$$

The statement of the complementary virtual work principle is as follows: Assume that (a) given the displacement field $\tilde{\mathbf{u}}(\mathbf{x})$ $(\mathbf{x} \in A_u)$ and (b) the strain field $\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}^T(\mathbf{x})$ $(\mathbf{x} \in V)$ is a fixed one. Assume further that equation (7.67a) holds for any $\delta\boldsymbol{\sigma}$ under the side conditions (7.67b). Then the strain field $\boldsymbol{\varepsilon}$ is kinematically admissible.

The proof of this statement is different from the proof of the principle of virtual power since side condition $(7.67\text{b})_1$ can not be substituted directly into equation (7.67a). To avoid this difficulty we shall add the identically zero integral

$$\int_V \boldsymbol{\lambda} \cdot (\delta\boldsymbol{\sigma} \cdot \nabla)\, \mathrm{d}V = \mathbf{0}\,, \tag{7.68}$$

in which the arbitrary non zero vector field $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mathbf{x})$ is the Lagrange multiplier, to equation (7.67a). We get

$$\int_V [\boldsymbol{\varepsilon}\cdot\cdot\,\delta\boldsymbol{\sigma} + \boldsymbol{\lambda}\cdot(\delta\boldsymbol{\sigma}\cdot\nabla)]\,\mathrm{d}V = \int_{A_u} \tilde{\mathbf{u}}\cdot\delta\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}A\,. \tag{7.69}$$

It can also be checked with ease by taking the symmetry of $\delta\boldsymbol{\sigma}$ into account that the relation

$$\begin{aligned}\boldsymbol{\lambda}\cdot(\delta\boldsymbol{\sigma}\cdot\nabla) &= (\boldsymbol{\lambda}\cdot\delta\boldsymbol{\sigma})\cdot\nabla - \overset{\downarrow}{\boldsymbol{\lambda}}\cdot\delta\boldsymbol{\sigma}\cdot\nabla = (\boldsymbol{\lambda}\cdot\delta\boldsymbol{\sigma})\cdot\nabla - (\boldsymbol{\lambda}\circ\nabla)\cdot\cdot\,\delta\boldsymbol{\sigma} = \\ &= (\boldsymbol{\lambda}\cdot\delta\boldsymbol{\sigma})\cdot\nabla - \frac{1}{2}(\boldsymbol{\lambda}\circ\nabla + \nabla\circ\boldsymbol{\lambda})\cdot\cdot\,\delta\boldsymbol{\sigma}\end{aligned} \tag{7.70}$$

is an identity. Since

$$\int_V (\boldsymbol{\lambda}\cdot\delta\boldsymbol{\sigma})\cdot\nabla\mathrm{d}V = \int_{A=A_t\cup A_u} \boldsymbol{\lambda}\cdot\delta\boldsymbol{\sigma}\cdot\mathbf{n}\mathrm{d}A = \underset{\delta\boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{0}\ \mathbf{x}\in A_t}{\uparrow} = \int_{A_u} \boldsymbol{\lambda}\cdot\delta\boldsymbol{\sigma}\cdot\mathbf{n}\mathrm{d}A \tag{7.71}$$

substituting (7.70) into (7.69) yields

$$\int_V \left[\boldsymbol{\varepsilon} - \frac{1}{2}(\boldsymbol{\lambda}\circ\nabla + \nabla\circ\boldsymbol{\lambda})\right]\cdot\cdot\,\delta\boldsymbol{\sigma}\,\mathrm{d}V - \int_{A_u} [\tilde{\mathbf{u}} - \boldsymbol{\lambda}]\cdot\delta\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}A = \mathbf{0}\,. \tag{7.72}$$

Hence

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{\lambda}\circ\nabla + \nabla\circ\boldsymbol{\lambda})\,, \qquad \mathbf{x}\in V \tag{7.73}$$

and

$$\boldsymbol{\lambda} = \tilde{\mathbf{u}} \qquad \mathbf{x}\in A_u\,. \tag{7.74}$$

This means that $\boldsymbol{\varepsilon}$ is really kinematically admissible.

7.4.4. A solution algorithm for equilibrium problems. Two loading cases will be considered:

I. The body forces $\rho\mathbf{b} = \mathbf{f}$ and surface traction load $\lambda_A\tilde{\mathbf{t}} = \tilde{\mathbf{t}}^\circ$ are constants, i.e., they are independent of the deformation of the body (they are independent of the motion). This loading type is called dead load.

II. The body forces $\rho\mathbf{b} = \mathbf{f}$ are independent of the deformation of the body. In contrast to this the surface traction load has, however, the following form $\tilde{\mathbf{t}} = \tilde{p}\mathbf{n}$, where $\tilde{p}$ is constant. This surface traction load is referred to as follower load (or hydrostatc load).

The solution algorithm is grounded on the principle of virtual work which we shall regard in material description, i.e., in the initial configuration of the body.

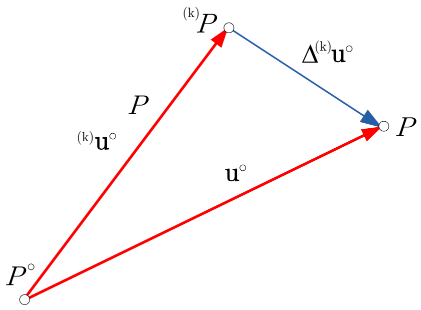

FIGURE 7.1. Displacement vectors for two successive iteration steps

In accordance with Figure 7.1 we shall assume that the displacement field in the k-th state of the continuum is ${}^{(k)}\mathbf{u}^\circ$. The increment $\Delta^{(k)}\mathbf{u}^\circ$ of the displacement field makes possible to get into the next state of the continuum. For the sake of simplicity in our notations we shall not apply a separate notation, i.e., the prefix $(k+1)$ to the quantities that describe the next state of the continuum. Thus, for instance:

$$\mathbf{u}^\circ = {}^{(k)}\mathbf{u}^\circ + \Delta^{(k)}\mathbf{u}^\circ \,. \tag{7.75}$$

After having the k-th iteration step performed the physical and geometrical quantities ${}^{(k)}\mathbf{u}^\circ$, ${}^{(k)}\boldsymbol{E}$, ${}^{(k)}\boldsymbol{S}$, and ${}^{(k)}\lambda_V$ ${}^{(k)}\lambda_A$ are all known. Substituting $\mathbf{u}^\circ$ into equation (2.39) yields

$$\boldsymbol{E} = {}^{(k)}\boldsymbol{E} + \Delta^{(k)}\boldsymbol{E} = \frac{1}{2}\left[\left({}^{(k)}\mathbf{u}^\circ + \Delta^{(k)}\mathbf{u}^\circ\right)\circ\nabla^\circ + \nabla^\circ\circ\left({}^{(k)}\mathbf{u}^\circ + \Delta^{(k)}\mathbf{u}^\circ\right)\right] + \\ + \frac{1}{2}\left[\nabla^\circ\circ\left({}^{(k)}\mathbf{u}^\circ + \Delta^{(k)}\mathbf{u}^\circ\right)\right]\cdot\left[\left({}^{(k)}\mathbf{u}^\circ + \Delta^{(k)}\mathbf{u}^\circ\right)\circ\nabla^\circ\right] = {}^{(k)}\boldsymbol{E} + \Delta^{(k)}\boldsymbol{E}^L + \Delta^{(k)}\boldsymbol{E}^N, \tag{7.76}$$

for the Green-Lagrange strain tensor, where

$$\Delta^{(k)}\boldsymbol{E}^L = \frac{1}{2}\Big\{\left[\left(\Delta^{(k)}\mathbf{u}^\circ\right)\circ\nabla^\circ + \nabla^\circ\circ\left(\Delta^{(k)}\mathbf{u}^\circ\right)\right] + \\ + \frac{1}{2}\left[\nabla^\circ\circ\left(\Delta^{(k)}\mathbf{u}^\circ\right)\right]\cdot\left[{}^{(k)}\mathbf{u}^\circ\circ\nabla^\circ\right] + \left[\nabla^\circ\circ{}^{(k)}\mathbf{u}^\circ\right]\cdot\left[\left(\Delta^{(k)}\mathbf{u}^\circ\right)\circ\nabla^\circ\right]\Big\} \tag{7.77a}$$

and

$$\Delta^{(k)}\boldsymbol{E}^N = \frac{1}{2}\left[\nabla^\circ\circ\left(\Delta^{(k)}\mathbf{u}^\circ\right)\right]\cdot\left[\left(\Delta^{(k)}\mathbf{u}^\circ\right)\circ\nabla^\circ\right] \tag{7.77b}$$

are the parts of the increment $\Delta^{(k)}\boldsymbol{E}$ being linear and non-linear in $\Delta^{(k)}\mathbf{u}^\circ$. As regards the second Piola-Kirchhoff stress tensor we can write

$$\boldsymbol{S} = {}^{(k)}\boldsymbol{S} + \Delta^{(k)}\boldsymbol{S} \,. \tag{7.78}$$

We shall assume that the virtual displacement field is given by

$$\delta\mathbf{u}^\circ = \delta\left(\Delta^{(k)}\mathbf{u}^\circ\right) \tag{7.79}$$

which means that we do not change the displacement field ${}^{(k)}\mathbf{u}^\circ$. Then

$$\delta \boldsymbol{E} = \delta\left(\Delta^{(k)}\boldsymbol{E}\right) = \delta\left(\Delta^{(k)}\boldsymbol{E}^L\right) + \delta\left(\Delta^{(k)}\boldsymbol{E}^N\right) \tag{7.80}$$

is the virtual strain field in which

$$\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right) = \frac{1}{2}\left[\delta\mathbf{u}^\circ \circ \nabla^\circ + \nabla^\circ \circ \delta\mathbf{u}^\circ\right] + \\ + \frac{1}{2}\left\{\left[\nabla^\circ \circ \delta\mathbf{u}^\circ\right] \cdot \left[{}^{(k)}\mathbf{u}^\circ \circ \nabla^\circ\right] + \left[\nabla^\circ \circ {}^{(k)}\mathbf{u}^\circ\right] \cdot \left[\delta\mathbf{u}^\circ \circ \nabla^\circ\right]\right\} \tag{7.81a}$$

and

$$\delta\left(\Delta^{(k)}\boldsymbol{E}^N\right) = \frac{1}{2}\left\{\left[\nabla^\circ \circ \delta\mathbf{u}^\circ\right] \cdot \left[\Delta^{(k)}\mathbf{u}^\circ \circ \nabla^\circ\right] + \left[\nabla^\circ \circ \Delta^{(k)}\mathbf{u}^\circ\right] \cdot \left[\delta\mathbf{u}^\circ \circ \nabla^\circ\right]\right\}. \tag{7.81b}$$

Note that $\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right)$ does not contain $\Delta^{(k)}\mathbf{u}^\circ$. In contrast to this $\delta\left(\Delta^{(k)}\boldsymbol{E}^N\right)$ is a homogeneous linear function of $\Delta^{(k)}\mathbf{u}^\circ$.

The equations that are valid for the area element ratio λ_A and the volume element ration λ_V have the same structure as equations (7.75), (7.76) and (7.78):

$$\lambda_A = {}^{(k)}\lambda_A + \Delta^{(k)}\lambda_A\,, \qquad \lambda_V = {}^{(k)}\lambda_V + \Delta^{(k)}\lambda_V\,. \tag{7.82}$$

Now it is our main objective to clarify how the increments $\Delta^{(k)}\lambda_A$ and $\Delta^{(k)}\lambda_V$ are related to the displacement gradient $\Delta^{(k)}\mathbf{u}\circ\nabla^\circ$.

To begin with we shall consider the surface element vector given by equation (2.89):

$$\mathrm{d}\mathbf{A} = J\,\boldsymbol{F}^{-T}\cdot \mathrm{d}\mathbf{A}^\circ = J\,\mathrm{d}\mathbf{A}^\circ \cdot \boldsymbol{F}^{-1} \underset{(2.27)}{=} \mathrm{d}\mathbf{A}^\circ \cdot \boldsymbol{\mathcal{F}} \tag{7.83}$$

Hence

$$\Delta^{(k)}\mathrm{d}\mathbf{A} = \mathrm{d}\mathbf{A}^\circ \cdot \Delta^{(k)}\boldsymbol{\mathcal{F}}. \tag{7.84}$$

According to (2.26)

$$\mathcal{F}_{PL} = \frac{1}{2}e_{PQR}e_{LJK}\left(\delta_{JQ} + u^\circ_{J,Q}\right)\left(\delta_{KR} + u^\circ_{K,R}\right) \tag{7.85}$$

which means that

$$\begin{aligned}
\mathcal{F}_{PL} &= {}^{(k)}\mathcal{F}_{PL} + \Delta^{(k)}\mathcal{F}_{PL} = \\
&= \frac{1}{2}e_{PQR}e_{LJK}\left(\delta_{JQ} + u^\circ_{J,Q} + \Delta^{(k)}u^\circ_{J,Q}\right)\left(\delta_{KR} + u^\circ_{K,R} + \Delta^{(k)}u^\circ_{K,R}\right) = \\
&= {}^{(k)}\mathcal{F}_{PL} + \frac{1}{2}e_{PQR}e_{LJK}\,\Delta^{(k)}u^\circ_{J,Q}\left(\delta_{KR} + u^\circ_{K,R}\right) + \\
&\qquad + \frac{1}{2}e_{PQR}e_{LJK}\,\Delta^{(k)}u^\circ_{K,R}\left(\delta_{JQ} + u^\circ_{J,Q}\right) + \\
&+ \frac{1}{2}e_{PQR}e_{LJK}\,\Delta^{(k)}u^\circ_{J,Q}\Delta^{(k)}u^\circ_{K,R} = {}^{(k)}\mathcal{F}_{PL} + \\
&+ e_{PQR}e_{LJK}\left(\delta_{JQ} + u^\circ_{J,Q}\right)\Delta^{(k)}u^\circ_{K,R} + \frac{1}{2}e_{PQR}e_{LJK}\,\Delta^{(k)}u^\circ_{J,Q}\Delta^{(k)}u^\circ_{K,R}\,.
\end{aligned}$$

Consequently,

$$\Delta^{(k)}\mathcal{F}_{PL} = \underbrace{e_{PQR}e_{LJK}\ \left(\delta_{JQ} + u^{\circ}_{J,Q}\right)\Delta^{(k)}u^{\circ}_{K,R}}_{\Delta^{(k)}\mathcal{F}^{(1)}_{PL}} + \\ + \underbrace{\frac{1}{2}e_{PQR}e_{LJK}\ \Delta u^{\circ}_{J,Q}\Delta u^{\circ}_{K,R}}_{\Delta^{(k)}\mathcal{F}^{(2)}_{PL}} \tag{7.86}$$

and

$$\Delta^{(k)}\mathrm{d}A_L = \mathrm{d}A^{\circ}_P\,\Delta^{(k)}\mathcal{F}_{PL} = \underbrace{\mathrm{d}A^{\circ}_P\,\Delta^{(k)}\mathcal{F}^{(1)}_{PL}}_{\Delta^{(k)}\mathrm{d}A^{(1)}_L} + \underbrace{\mathrm{d}A^{\circ}_P\,\Delta^{(k)}\mathcal{F}^{(2)}_{PL}}_{\Delta^{(k)}\mathrm{d}A^{(2)}_L}. \tag{7.87}$$

Note that $[\Delta^{(k)}\mathrm{d}A^{(1)}_L]$ $\{\ \Delta^{(k)}\mathrm{d}A^{(2)}_L\}$ is a homogeneous [linear] {quadratic} function of the displacement gradient $\Delta^{(k)}u^{\circ}_{M,N}$.

Since

$$\mathrm{d}A^{\circ}_P = n^{\circ}_P\mathrm{d}A^{\circ} \tag{7.88}$$

it also holds that

$$\mathrm{d}A^{\circ}_P\,\Delta^{(k)}\mathcal{F}^{(1)}_{PL} = n^{\circ}_P\Delta^{(k)}\mathcal{F}^{(1)}_{PL}\mathrm{d}A = e_{LJK}n^{\circ}_P e_{PQR}\,F_{JQ}\Delta^{(k)}u^{\circ}_{K,R}\mathrm{d}A = \\ = e_{LJK}\left[n^{\circ}_P e_{PQR}\,F_{JQ}\nabla^{\circ}_R\right]\Delta^{(k)}u^{\circ}_K\mathrm{d}A = D_{LK}\Delta^{(k)}u^{\circ}_K\mathrm{d}A\,, \tag{7.89}$$

where

$$D_{LK} = e_{LJK}\left[n^{\circ}_P e_{PQR}\,F_{JQ}\nabla^{\circ}_R\right]. \tag{7.90}$$

With (7.87) and (7.88) we get

$$\mathrm{d}A_L = n^{\circ}_P\left({}^{(k)}\mathcal{F}_{PL} + \Delta^{(k)}\mathcal{F}^{(1)}_{PL} + \Delta^{(k)}\mathcal{F}^{(2)}_{PL}\right). \tag{7.91}$$

Now we shall proceed with the scalar surface element. Making use of equations (2.90a) and (2.27) we may write

$$\lambda^2_A = \frac{\mathrm{d}A^2}{(\mathrm{d}A^{\circ})^2} = \mathbf{n}^{\circ}\cdot\boldsymbol{\mathcal{F}}\cdot\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ}\,. \tag{7.92}$$

Thus

$$\left({}^{(k)}\lambda_A + \Delta^{(k)}\lambda_A\right)^2 = \frac{\mathrm{d}A^2}{(\mathrm{d}A^{\circ})^2} = \\ = \mathbf{n}^{\circ}\cdot\left({}^{(k)}\boldsymbol{\mathcal{F}} + \Delta^{(k)}\boldsymbol{\mathcal{F}}\right)\cdot\left({}^{(k)}\boldsymbol{\mathcal{F}}^T + \Delta^{(k)}\boldsymbol{\mathcal{F}}^T\right)\cdot\mathbf{n}^{\circ} = {}^{(k)}\lambda^2_A + \\ + \mathbf{n}^{\circ}\cdot{}^{(k)}\boldsymbol{\mathcal{F}}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ} + \mathbf{n}^{\circ}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}\cdot{}^{(k)}\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ} + \mathbf{n}^{\circ}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ}\,,$$

from where

$${}^{(k)}\lambda_A + \Delta^{(k)}\lambda_A = \\ = {}^{(k)}\lambda_A\sqrt{1 + \frac{1}{{}^{(k)}\lambda^2_A}\left(\mathbf{n}^{\circ}\cdot{}^{(k)}\boldsymbol{\mathcal{F}}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ} + \mathbf{n}^{\circ}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}\cdot{}^{(k)}\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ} + \mathbf{n}^{\circ}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ}\right)} \simeq \\ \simeq {}^{(k)}\lambda_A + \frac{1}{2\,{}^{(k)}\lambda^2_A}\left(\mathbf{n}^{\circ}\cdot{}^{(k)}\boldsymbol{\mathcal{F}}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ} + \mathbf{n}^{\circ}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}\cdot{}^{(k)}\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ} + \mathbf{n}^{\circ}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^T\cdot\mathbf{n}^{\circ}\right).$$

Here it has been taken into account that the second term under the square root sign is much smaller than one ($\sqrt{1+x} \simeq 1 + x/2$ if $|x| \ll 1$). Consequently,

$$\Delta^{(k)}\lambda_A = \underbrace{\frac{1}{\lambda_A}\mathbf{n}^\circ \cdot \Delta^{(k)}\boldsymbol{\mathcal{F}} \cdot {}^{(k)}\boldsymbol{\mathcal{F}}^T \cdot \mathbf{n}^\circ}_{\Delta^{(k)}\lambda_A^{(1)}} + \underbrace{\frac{1}{2\lambda_A}\mathbf{n}^\circ \cdot \Delta^{(k)}\boldsymbol{\mathcal{F}} \cdot \Delta^{(k)}\boldsymbol{\mathcal{F}}^T \cdot \mathbf{n}^\circ}_{\Delta^{(k)}\lambda_A^{(2)}} \tag{7.93}$$

We remark that $[\Delta^{(k)}\lambda_A^{(1)}]$ $\{\ \Delta^{(k)}\lambda_A^{(2)}\}$ is a homogenous [linear] {quadratic} function of the displacement gradient $\Delta^{(k)}u^\circ_{M,N}$.

We shall proceed with λ_V. According to equation (2.24)$_2$

$$\Delta^{(k)}F_{AB} = \Delta^{(k)}u^\circ_{A,B}$$

and hence

$$\lambda_V = \frac{\mathrm{d}V}{\mathrm{d}V^\circ} = J = \left|{}^{(k)}F_{AB} + \Delta^{(k)}F_{AB}\right| = \underset{(1.50)}{\uparrow} =$$
$$= \frac{1}{6}e_{IJK}e_{PQR}\left({}^{(k)}F_{IP}+\Delta^{(k)}u^\circ_{I,P}\right)\left({}^{(k)}F_{JQ}+\Delta^{(k)}u^\circ_{J,Q}\right)\left({}^{(k)}F_{KR}+\Delta^{(k)}u^\circ_{K,R}\right),$$

from where we get

$$\Delta\lambda_V = \underbrace{\frac{1}{2}e_{IJK}e_{PQR}F_{IP}F_{JQ}\Delta^{(k)}u^\circ_{K,R}}_{\Delta^{(k)}\lambda_V^{(1)}} + \underbrace{\frac{1}{2}e_{IJK}e_{PQR}F_{IP}\Delta^{(k)}u^\circ_{J,Q}\Delta^{(k)}u^\circ_{K,R}}_{\Delta^{(k)}\lambda_V^{(2)}} +$$
$$+ \underbrace{\frac{1}{6}e_{IJK}e_{PQR}\Delta^{(k)}u^\circ_{I,P}\Delta^{(k)}u^\circ_{J,Q}\Delta^{(k)}\Delta u^\circ_{K,R}}_{\Delta\lambda^{(k)}{}_V^{(3)}} \tag{7.94}$$

Note that $\Delta\lambda_V^{(1)}$ is a homogeneous linear function of $\Delta^{(k)}u^\circ_K$, while $\Delta\lambda_V^{(2)}$ and $\Delta\lambda_V^{(3)}$ are homogeneous quadratic and cubic functions of the displacement gradient $\Delta^{(k)}u^\circ_{M,N}$.

For equilibrium problems $\mathbf{a} = \mathbf{0}$. Thus

$$\mathbf{f}^{(a)}\,\mathrm{d}V = \mathbf{f}\,\mathrm{d}V = \rho\mathbf{b}\,\mathrm{d}V = \lambda_V\,\rho\mathbf{b}\,\mathrm{d}V^\circ = \rho^\circ\mathbf{b}^\circ\,\mathrm{d}V^\circ.$$

For Loads I. and II. the principle of virtual work (7.60) assumes the following forms:

$$\int_{V^\circ}\boldsymbol{S}\cdot\cdot\,\delta\boldsymbol{E}\,\mathrm{d}V^\circ = \int_{V^\circ}\lambda_V\,\rho\mathbf{b}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}V^\circ + \int_{A_t^\circ}\lambda_A\,\tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}A^\circ\,. \tag{7.95a}$$

$$\int_{V^\circ}\boldsymbol{S}\cdot\cdot\,\delta\boldsymbol{E}\,\mathrm{d}V^\circ = \int_{V^\circ}\lambda_V\,\rho\mathbf{b}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}V^\circ + \int_{A_t^\circ}\tilde{t}\,\delta\mathbf{u}^\circ\cdot\mathbf{n}\,\lambda_A\,\mathrm{d}A^\circ\,. \tag{7.95b}$$

After substituting equations (7.78), (7.80) and (7.82) into equation (7.95a) we get

$$\int_{V^\circ} \left({}^{(k)}\boldsymbol{S} + \Delta^{(k)}\boldsymbol{S}\right) \cdot\cdot \left[\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right) + \delta\left(\Delta^{(k)}\boldsymbol{E}^N\right)\right] \mathrm{d}V^\circ = \\ = \int_{V^\circ} \left({}^{(k)}\lambda_V + \Delta^{(k)}\lambda_V\right) \rho\mathbf{b}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}V^\circ + \int_{A_t^\circ} \left({}^{(k)}\lambda_A + \Delta^{(k)}\lambda_A\right) \tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}A^\circ . \tag{7.96}$$

or

$$\int_{V^\circ} \left({}^{(k)}\boldsymbol{S} + \Delta^{(k)}\boldsymbol{S}\right) \cdot\cdot \left[\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right) + \delta\left(\Delta^{(k)}\boldsymbol{E}^N\right)\right] \mathrm{d}V^\circ = \\ = \int_{V^\circ} \left({}^{(k)}\lambda_V + \Delta^{(k)}\lambda_V^{(1)} + \Delta^{(k)}\lambda_V^{(2)} + \Delta^{(k)}\lambda_V^{(3)}\right) \rho\mathbf{b}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}V^\circ + \\ + \int_{A_t^\circ} \left({}^{(k)}\lambda_A + \Delta^{(k)}\lambda_A^{(1)} + \Delta^{(k)}\lambda_A^{(2)}\right) \tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}A^\circ . \tag{7.97}$$

if we take (7.94) and (7.87) also into account. A rearrangement separates the linear and nonlinear terms in this equation:

$$\int_{V^\circ} \Delta^{(k)}\boldsymbol{S} \cdot\cdot\, \delta\left(\Delta^{(k)}\boldsymbol{E}^L\right) \mathrm{d}V^\circ + \int_{V^\circ} {}^{(k)}\boldsymbol{S} \cdot\cdot\, \delta\left(\Delta^{(k)}\boldsymbol{E}^N\right) \mathrm{d}V^\circ + \\ - \int_{V^\circ} \Delta^{(k)}\lambda_V^{(1)}\, \rho\mathbf{b}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}V^\circ - \int_{A_t^\circ} \Delta^{(k)}\lambda_A^{(1)}\, \tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}A^\circ + \\ + \underbrace{\int_{V^\circ} \Delta^{(k)}\boldsymbol{S} \cdot\cdot\, \delta\left(\Delta^{(k)}\boldsymbol{E}^N\right) \mathrm{d}V^\circ - \int_{V^\circ} \left(\Delta^{(k)}\lambda_V^{(2)} + \Delta^{(k)}\lambda_V^{(3)}\right) \rho\mathbf{b}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}V^\circ}_{\text{non-linear in } \Delta^{(k)}u^\circ_{M,N}} = \\ = - \int_{V^\circ} {}^{(k)}\boldsymbol{S} \cdot\cdot\, \delta\left(\Delta^{(k)}\boldsymbol{E}^L\right) \mathrm{d}V^\circ + \int_{V^\circ} {}^{(k)}\lambda_V\, \rho\mathbf{b}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}V^\circ + \\ + \int_{A_t^\circ} {}^{(k)}\lambda_A\, \tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}A^\circ + \underbrace{\int_{A_t^\circ} \Delta^{(k)}\lambda_A^{(2)}\, \tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}A^\circ}_{\text{non-linear in } \Delta^{(k)}u^\circ_{M,N}} . \tag{7.98}$$

If we drop the non-linear terms we obtain a linearized equation:

$$\boxed{\begin{aligned} &\int_{V^\circ} \Delta^{(k)}\boldsymbol{S} \cdot\cdot\, \delta\left(\Delta^{(k)}\boldsymbol{E}^L\right) \mathrm{d}V^\circ + \int_{V^\circ} {}^{(k)}\boldsymbol{S} \cdot\cdot\, \delta\left(\Delta^{(k)}\boldsymbol{E}^N\right) \mathrm{d}V^\circ + \\ &\quad - \int_{V^\circ} \Delta^{(k)}\lambda_V^{(1)}\, \rho\mathbf{b}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}V^\circ - \int_{A_t^\circ} \Delta^{(k)}\lambda_A^{(1)}\, \tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}A^\circ = \\ &= - \int_{V^\circ} {}^{(k)}\boldsymbol{S} \cdot\cdot\, \delta\left(\Delta^{(k)}\boldsymbol{E}^L\right) \mathrm{d}V^\circ + \int_{V^\circ} {}^{(k)}\lambda_V\, \rho\mathbf{b}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}V^\circ + \\ &\quad + \int_{A_t^\circ} {}^{(k)}\lambda_A\, \tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ \,\mathrm{d}A^\circ . \end{aligned}} \tag{7.99}$$

As regards the case of follower loads the integral taken on A_t° is different:

$$\int_{A_t^\circ} \tilde{t}\,\delta\mathbf{u}^\circ\cdot\mathbf{n}\lambda_A\,\mathrm{d}A^\circ = \underset{(7.91)}{\uparrow} = \\ = \int_{A_t^\circ} \tilde{t}\,\mathbf{n}^\circ\cdot\left({}^{(k)}\boldsymbol{\mathcal{F}} + \Delta^{(k)}\boldsymbol{\mathcal{F}}^{(1)} + \Delta^{(k)}\boldsymbol{\mathcal{F}}^{(2)}\right)\cdot\delta\mathbf{u}^\circ\mathrm{d}A^\circ \quad (7.100)$$

Substituting this relationship for the integral taken on A_t° in equation (7.97), the following result is obtained:

$$\int_{V^\circ}\Delta^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right)\mathrm{d}V^\circ + \int_{V^\circ}{}^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^N\right)\mathrm{d}V^\circ + \\ -\int_{V^\circ}\Delta^{(k)}\lambda_V^{(1)}\,\rho\mathbf{b}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}V^\circ - \int_{A_t^\circ}\tilde{t}\,\mathbf{n}^\circ\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^{(1)}\cdot\delta\mathbf{u}^\circ\mathrm{d}A^\circ + \\ + \underbrace{\int_{V^\circ}\Delta^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^N\right)\mathrm{d}V^\circ - \int_{V^\circ}\left(\Delta^{(k)}\lambda_V^{(2)} + \Delta^{(k)}\lambda_V^{(3)}\right)\rho\mathbf{b}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}V^\circ}_{\text{non-linear in } \Delta^{(k)}u^\circ_{M,N}} = \\ = -\int_{V^\circ}{}^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right)\mathrm{d}V^\circ + \int_{V^\circ}{}^{(k)}\lambda_V\,\rho\mathbf{b}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}V^\circ + \\ + \int_{A_t^\circ}\tilde{t}\,\mathbf{n}^\circ\cdot{}^{(k)}\boldsymbol{\mathcal{F}}\cdot\delta\mathbf{u}^\circ\mathrm{d}A^\circ + \underbrace{\int_{A_t^\circ}\tilde{t}\,\mathbf{n}^\circ\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^{(2)}\cdot\delta\mathbf{u}^\circ\mathrm{d}A^\circ}_{\text{non-linear in } \Delta^{(k)}u^\circ_{M,N}}. \quad (7.101)$$

Neglecting the non-linear terms in this equation yields the following result for Load II:

$$\boxed{\begin{aligned} &\int_{V^\circ}\Delta^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right)\mathrm{d}V^\circ + \int_{V^\circ}{}^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^N\right)\mathrm{d}V^\circ + \\ &\quad -\int_{V^\circ}\Delta^{(k)}\lambda_V^{(1)}\,\rho\mathbf{b}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}V^\circ - \int_{A_t^\circ}\tilde{t}\,\mathbf{n}^\circ\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^{(1)}\cdot\delta\mathbf{u}^\circ\mathrm{d}A^\circ = \\ &= -\int_{V^\circ}{}^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right)\mathrm{d}V^\circ + \int_{V^\circ}{}^{(k)}\lambda_V\,\rho\mathbf{b}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}V^\circ + \\ &\quad + \int_{A_t^\circ}\tilde{t}\,\mathbf{n}^\circ\cdot{}^{(k)}\boldsymbol{\mathcal{F}}\cdot\delta\mathbf{u}^\circ\mathrm{d}A^\circ \end{aligned}} \quad (7.102)$$

REMARK 7.11: Suppose that it is not the body forces per unit volume $\mathbf{f} = \rho\mathbf{b}$ but the body forces per unit mass are constants, i.e., it holds that $\mathbf{b} = \mathbf{b}^\circ$. Then

$\rho\mathbf{b}\,\mathrm{d}V = \rho^\circ\mathbf{b}^\circ\,\mathrm{d}V^\circ$. Consequently, equations (7.99) and (7.102) simplify to:

$$\boxed{\begin{aligned}&\int_{V^\circ}\Delta^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right)\mathrm{d}V^\circ+\int_{V^\circ}{}^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^N\right)\mathrm{d}V^\circ+\\&\quad-\int_{A_t^\circ}\Delta^{(k)}\lambda_A^{(1)}\,\tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}A^\circ=-\int_{V^\circ}{}^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right)\mathrm{d}V^\circ+\\&\quad+\int_{V^\circ}\rho^\circ\mathbf{b}^\circ\cdot\delta\mathbf{u}^\circ\,\mathrm{d}V^\circ+\int_{A_t^\circ}{}^{(k)}\lambda_A\,\tilde{\mathbf{t}}\cdot\delta\mathbf{u}^\circ\,\mathrm{d}A^\circ\end{aligned}}\tag{7.103}$$

and

$$\boxed{\begin{aligned}&\int_{V^\circ}\Delta^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right)\mathrm{d}V^\circ+\int_{V^\circ}{}^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^N\right)\mathrm{d}V^\circ+\\&\quad-\int_{A_t^\circ}\tilde{t}\,\mathbf{n}^\circ\cdot\Delta^{(k)}\boldsymbol{\mathcal{F}}^{(1)}\cdot\delta\mathbf{u}^\circ\mathrm{d}A^\circ=-\int_{V^\circ}{}^{(k)}\boldsymbol{S}\cdot\cdot\,\delta\left(\Delta^{(k)}\boldsymbol{E}^L\right)\mathrm{d}V^\circ+\\&\quad+\int_{V^\circ}\rho^\circ\mathbf{b}^\circ\cdot\delta\mathbf{u}^\circ\,\mathrm{d}V^\circ+\int_{A_t^\circ}\tilde{t}\,\mathbf{n}^\circ\cdot{}^{(k)}\boldsymbol{\mathcal{F}}\cdot\delta\mathbf{u}^\circ\mathrm{d}A^\circ\,.\end{aligned}}\tag{7.104}$$

The proof of equations (7.103) and (7.104) is left for Problem 7.2.

Due to the fact that the non-linear terms are neglected the principle of virtual work (7.99) (or (7.102)) does not related to an equilibrium state of the body. Hence the solution (the unknown equilibrium state) can be determined by performing a series of iteration steps. The iteration procedure should be supplemented by appropriate error limits and a constitutive equation. If the body is hyperelastic for instance then

$$\begin{aligned}\Delta^{(k)}S_{AB}&=\left.\frac{\partial S_{AB}}{\partial E_{MN}}\right|_{^{(k)}\boldsymbol{E}}\Delta E_{MN}=\\&=\rho^\circ\left.\frac{\partial^2 e}{\partial E_{AB}\partial E_{MN}}\right|_{^{(k)}\boldsymbol{E}}\Delta E_{MN}\approx\rho^\circ\left.\frac{\partial^2 e}{\partial E_{AB}\partial E_{MN}}\right|_{^{(k)}\boldsymbol{E}}\Delta E^L_{MN}\,.\end{aligned}\tag{7.105}$$

As regards the derivative $\frac{\partial S_{AB}}{\partial E_{MN}}$ we refer the reader to equaton (8.134).

In the iteration procedure the k^{th} state is regarded as a known state before performing the $k+1$ iteration step.

In the first step of the iteration the fields ${}^{(1)}\mathbf{u}^\circ$, ${}^{(1)}\boldsymbol{E}$ and ${}^{(1)}\boldsymbol{S}$ should be known quantities – we have to select them in some way – while the deformation measures ${}^{(1)}\lambda_V$ and ${}^{(1)}\lambda_A$ are set to 1. The displacement increment $\Delta^{(1)}\mathbf{u}^\circ$ is the unknown.

In the k^{th} step of the iteration the displacement increment $\Delta^{(k)}\mathbf{u}^\circ$ is the unknown and the increment of the stress tensor $\Delta^{(k)}\boldsymbol{S}$ is computed with linear approximation from the strain tensor increment $\Delta^{(k)}\boldsymbol{E}^L$.

After performing the k^{th} iteration step

$$^{(k+1)}\mathbf{u}^{\circ} = {}^{(k)}\mathbf{u}^{\circ} + \Delta^{(k)}\mathbf{u}^{\circ}\,, \tag{7.106a}$$

$$^{(k+1)}\boldsymbol{E} = {}^{(k)}\boldsymbol{E} + \Delta^{(k)}\boldsymbol{E} = {}^{(k)}\boldsymbol{E} + \Delta^{(k)}\boldsymbol{E}^{L} + \Delta^{(k)}\boldsymbol{E}^{N}\,, \tag{7.106b}$$

$$^{(k+1)}\boldsymbol{S} = {}^{(k)}\boldsymbol{S} + \Delta^{(k+1)}\boldsymbol{S}\,. \tag{7.106c}$$

7.5. Problems

PROBLEM 7.1: Prove the principle of complementary virtual power.

PROBLEM 7.2: Prove the validity of equations (7.103) and (7.104).

CHAPTER 8

Constitutive equations

8.1. Equations and variables in continuum mechanics

For given body forces $\rho\mathbf{b}$ and heat source distribution h the fundamental laws of the continuum mechanics in spatial description, i.e.,

– the continuity equation (6.5a) (or principle of mass conservation in local form):

$$(\rho)^{\boldsymbol{\cdot}} + \rho\,(\mathbf{v}\cdot\nabla) = 0\,, \qquad (\rho)^{\boldsymbol{\cdot}} + \rho\, v_{\ell,\ell} = 0\,; \tag{8.1}$$

– the kinematic equations (8.2a) (if the velocity field $\mathbf{v}$ is the unknown) or (8.2b) (if the displacement field $\mathbf{u}$ is the unknown):

$$\boldsymbol{d} = \frac{1}{2}\left(\mathbf{v}\circ\nabla + \nabla\circ\mathbf{v}\right)\,, \qquad d_{k\ell} = \frac{1}{2}\left(v_{k,\ell} + v_{\ell,k}\right)\,; \tag{8.2a}$$

or

$$\begin{aligned} \boldsymbol{e} &= \frac{1}{2}\left[\mathbf{u}\circ\nabla + \nabla\circ\mathbf{u} - (\nabla\circ\mathbf{u})\cdot(\mathbf{u}\circ\nabla)\right]\,, \\ e_{k\ell} &= \frac{1}{2}\left[u_{k,\ell} + u_{\ell,k} - u_{m,k}u_{m,\ell}\right]\,; \end{aligned} \tag{8.2b}$$

– Cauchy's equation of motion (6.13):

$$\boldsymbol{t}\cdot\nabla + \rho\mathbf{b} = \rho\mathbf{a}\,, \qquad t_{k\ell,\ell} + \rho\, b_k = \rho\, a_k \tag{8.3}$$

– and the first theorem of thermodynamics (6.53) in local form:

$$\rho\,(e)^{\boldsymbol{\cdot}} = \boldsymbol{t}\cdot\cdot\,\boldsymbol{d} + \rho h - \mathbf{q}\cdot\nabla\,, \qquad \rho\,(e)^{\boldsymbol{\cdot}} = t_{k\ell}d_{k\ell} + \rho h - q_{k,\ell} \tag{8.4}$$

contain the following scalar unknowns:

– the density ρ:	(1 scalar),
– {the velocity field $\mathbf{v}$:	(3 scalars),
– the strain rate tensor $\boldsymbol{d}$:	(6 scalar – symmetry)},
– [or the displacment field $\mathbf{u}$:	(3 scalars),
– the Euler-Almansi strain tensor $\boldsymbol{e}$	(6 scalars – symmetry)],
– the Cauchy stress tensor $\boldsymbol{t}$:	(6 scalars – symmetry),
– the internal energy e:	(1 scalar).

For solid bodies and isothermal case $h = 0$, $q_k = 0$ hence the number of scalar unknowns (ρ, {$\mathbf{v}$, $\boldsymbol{d}$}, [or $\mathbf{u}$, $\boldsymbol{e}$], $\boldsymbol{t}$ and e) is 17.

If the heat effects can not be neglected there are further unknowns which should also be determined. Namely:

– the temperature field Θ: (1 scalar),
– the entropy s (1 scalar),
– the heat flux $\mathbf{q}$: (3 scalar).

Consequently, the total number of unknowns is 22.

It is easy now to check that the number of scalar equations is as follows: (8.1) (1 scalar equation), (8.2a) or (8.2b) (6 scalar equations), (8.3) (3 scalar equations), (8.4) (1 scalar equation) which means that we have altogether 11 equations.

We can, therefore, come to the conclusion that

– 6 equations are missing if there are no heat effects,
– 11 equations are missing if the heat effects are to be taken into account.

The missing equations are called material equations or constitutive equations – the second expression is more general (the first expression is preferred if the heat effects can be neglected).

Remark 8.1: Note that the continuity equation (8.1) is solvable for the density ρ if the velocity field $\mathbf{v}$ is already known. With ρ the strain energy density e for a unit mass can be obtained from the first theorem of thermodynamics (8.4).

Assume that the fundamental equations are regarded in material description. For solid bodies and isothermal case with the displacement field as the fundamental unknown we have

– the continuity equation (6.8) again in local form:

$$\rho = \frac{1}{J}\rho^\circ; \tag{8.5}$$

– the kinematic equation (2.39) for the Green-Lagrange strain tensor:

$$\begin{gathered} \boldsymbol{E} = \frac{1}{2}\left[\mathbf{u}^\circ \circ \nabla^\circ + \nabla^\circ \circ \mathbf{u}^\circ + (\nabla^\circ \circ \mathbf{u}^\circ)\cdot(\mathbf{u}^\circ \circ \nabla^\circ)\right], \\ E_{AB} = \frac{1}{2}\left[u_{A,B} + u_{B,A} + u_{K,A}u_{K,B}\right]; \end{gathered} \tag{8.6}$$

– Cauchy's equation of motion (6.18):

$$(\boldsymbol{F}\cdot\boldsymbol{S})\cdot\nabla^\circ + \rho^\circ\,\mathbf{b}^\circ = \rho^\circ\,\mathbf{a}^\circ\,; \tag{8.7}$$

– and the first theorem of thermodynamics (6.64) in local form:

$$\rho^\circ\,(e)^{\bullet} = \boldsymbol{S}\cdot\cdot\,(\boldsymbol{E})^{\bullet}\,. \tag{8.8}$$

Equations (8.5), (8.6), (8.6) and (8.7), contain the following scalar unknowns:

– the density ρ: (1 scalar),
– the displacement field $\mathbf{u}^\circ$: (3 scalars),
– the Green-Lagrange strain tensor $\boldsymbol{E}$: (6 scalars – symmetry),
– the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$: (6 scalars – symmetry),
– the internal energy e: (1 scalar).

The number of scalar unknowns (ρ, $\{\mathbf{u}^\circ, \boldsymbol{E}\}$, $\boldsymbol{S}$ and e) is 17.

The number of scalar equations is as follows: (8.5) (1 scalar equation), (8.6) (6 scalar equations), (8.7) (3 scalar equations), (8.8) (1 scalar equation), which means that we have altogether 11 equations. Consequently, 6 equations are missing.

8.2. Some aspects of objectivity

8.2.1. Transformation and rotation.

8.2.1.1. *Transformation.* Subsection 1.2 is devoted to the issue of how to transform a vector from the unprimed coordinate system $(x_1\, x_2\, x_3)$ to the primed one system $(x'_1\, x'_2\, x'_3)$ and conversely from the primed coordinate system to the unprimed one. The transformation formulae (1.28) and (1.30) are presented in matrix form. By introducing the concept of the transformation tensor defined by the equation

$$\boldsymbol{\mathcal{T}} = \mathcal{T}_{k\ell'}\, \mathbf{i}_k \circ \mathbf{i}'_\ell\,, \qquad \mathcal{T}_{k\ell'} = Q_{\ell' k} \tag{8.9}$$

the matrix equations referred to above, i.e., equations (1.28) and (1.30) can be rewritten in tensorial form:

$$\mathbf{u}' = \boldsymbol{\mathcal{T}}^T \cdot \mathbf{u}\,, \qquad u'_\ell = \mathcal{T}_{\ell' k} u_k\,, \tag{8.10a}$$

$$\mathbf{u} = \boldsymbol{\mathcal{T}} \cdot \mathbf{u}'\,, \qquad u_k = \mathcal{T}_{k\ell'} u'_\ell\,. \tag{8.10b}$$

Tensorial equation {(8.10a)} [(8.10b)] is equivalent to the matrix equation {(1.28)} [(1.30)].

Remark 8.2: Remember that $\mathbf{u}'$ and $\mathbf{u}$ are the same vectors. This means that the tensor $\boldsymbol{\mathcal{T}}$ behaves as if it were the unit tensor, since equations (8.10) map the vector $\mathbf{u}$ onto itself.

It is obvious that

$$\underset{(3\times 3)}{\underline{\boldsymbol{\mathcal{T}}}} = \underset{(3\times 3)}{\underline{\mathbf{Q}}}^T \tag{8.11}$$

which means that the matrix $\underline{\boldsymbol{\mathcal{T}}}$ is the transpose of the matrix $\underline{\mathbf{Q}}$. By the determinant of $\boldsymbol{\mathcal{T}}$ we mean the determinant of its matrix: $|\mathcal{T}_{k\ell'}|$. Comparing equations (8.11), (1.31), (1.32) and recalling Remark 1.2 yield that the tensor $\boldsymbol{\mathcal{T}}$ satisfies the following relations:

$$\boldsymbol{\mathcal{T}}^T = \boldsymbol{\mathcal{T}}^{-1}\,, \qquad \det(\boldsymbol{\mathcal{T}}) = \det(\underset{(3\times 3)}{\underline{\boldsymbol{\mathcal{T}}}}) = 1\,. \tag{8.12}$$

Using the transformation tensor $\boldsymbol{\mathcal{T}}$ equations (1.73b) and (1.209) can also be rewritten:

$$\boldsymbol{W}' = \boldsymbol{\mathcal{T}}^T \cdot \boldsymbol{W} \cdot \boldsymbol{\mathcal{T}}\,, \qquad w'_{k\ell} = \mathcal{T}_{k'm} w_{mn} \mathcal{T}_{n\ell'}\,, \tag{8.13a}$$

$$\boldsymbol{W} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{W}' \cdot \boldsymbol{\mathcal{T}}^T\,, \qquad w_{mn} = \mathcal{T}_{mk'} w'_{k\ell} \mathcal{T}_{\ell' n}\,. \tag{8.13b}$$

Tensorial equations (8.13a) and (8.13b) are obviously equivalent to equations (1.73b) and (1.209).

Remark 8.3: Note that the tensors $\boldsymbol{W}$ and $\boldsymbol{W}'$ are the same. Therefore equations (8.13) show again that the tensor $\boldsymbol{\mathcal{T}}$ behaves as if it were a unit tensor.

The transformation tensor $\boldsymbol{\mathcal{T}}$ is, in general, a function of time. For our later considerations let us investigate the time derivative of the equation $\boldsymbol{\mathcal{T}} \cdot \boldsymbol{\mathcal{T}}^T = \boldsymbol{1}$. By applying the product rule we get

$$(\boldsymbol{\mathcal{T}})^{\bullet} \cdot \boldsymbol{\mathcal{T}}^T + \boldsymbol{\mathcal{T}} \cdot \left(\boldsymbol{\mathcal{T}}^T\right)^{\bullet} = \boldsymbol{O}\,.$$

Hence

$$(\boldsymbol{\mathcal{T}})^{\bullet} \cdot \boldsymbol{\mathcal{T}}^T = -\boldsymbol{\mathcal{T}} \cdot \left(\boldsymbol{\mathcal{T}}^T\right)^{\bullet} = -\left((\boldsymbol{\mathcal{T}})^{\bullet} \cdot \boldsymbol{\mathcal{T}}^T\right)^T\,, \tag{8.14}$$

which shows that $(\boldsymbol{\mathcal{T}})^{\bullet} \cdot \boldsymbol{\mathcal{T}}^T$ is a skew tensor. We remark that it is worth introducing the following notation:

$$(\boldsymbol{\mathcal{T}})^{\bullet} \cdot \boldsymbol{\mathcal{T}}^T = \hat{\Omega}_{k\ell}\, \mathbf{i}_k \circ \mathbf{i}_\ell\,; \qquad \hat{\Omega}_{11} = \hat{\Omega}_{22} = \hat{\Omega}_{33} = 0\,, \quad \hat{\Omega}_{k\ell} = -\hat{\Omega}_{\ell k} \quad \text{if} \quad k \neq \ell\,. \tag{8.15a}$$

The axial vector of $(\boldsymbol{\mathcal{T}})^{\bullet} \cdot \boldsymbol{\mathcal{T}}^T$ is given by the following relation:

$$\hat{\omega}_r = -\frac{1}{2} e_{k\ell r} \hat{\Omega}_{k\ell}\,. \tag{8.15b}$$

8.2.1.2. *Rotation.* If a vector, say the vector $\mathbf{u}$, is rotated together with the coordinate system – the rotated coordinate system is called primed coordinate system – then the components of the rotated vector $\overset{*}{\mathbf{u}}$ in the primed coordinate system will be the same as those in the unprimed coordinate system. This phenomenon is demonstrated graphically for the 2D vector $\mathbf{u}$ in Figure 8.1.

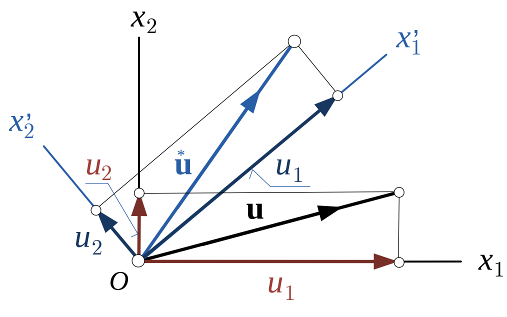

Figure 8.1. Rotated vector and coordinate system

Now we remind the reader of the fact that the unit vector $\mathbf{i}'_\ell$ can be given in terms of $\mathbf{i}_k$. According to equation (1.26c) we can write

$$\mathbf{i}'_\ell = Q_{\ell' k}\, \mathbf{i}_k\,, \qquad \text{or conversely} \qquad \mathbf{i}_k = Q_{k\ell'}\, \mathbf{i}'_\ell\,. \tag{8.16}$$

The rotated vector can, naturally, be given in the unprimed coordinate system provided that the above equation is taken into account. Substituting (8.16) into equation

$$\overset{*}{\mathbf{u}} = u_\ell\, \mathbf{i}'_\ell$$

yields

$$\begin{aligned}\overset{*}{\mathbf{u}} = u_\ell\, \mathbf{i}'_\ell = u_\ell\, Q_{\ell' k}\, \mathbf{i}_k = \mathbf{i}_k\, Q_{\ell' k} \underbrace{\delta_{\ell m}}_{\mathbf{i}_\ell \cdot \mathbf{i}_m} u_m = \\ = \big(\underbrace{Q_{\ell' k}\, \mathbf{i}_k \circ \mathbf{i}_\ell}_{\boldsymbol{Q}^T = \boldsymbol{\mathcal{R}}}\big) \cdot (u_m\, \mathbf{i}_m) = \boldsymbol{Q}^T \cdot \mathbf{u} = \boldsymbol{\mathcal{R}} \cdot \mathbf{u}\,,\end{aligned} \tag{8.17}$$

where (a) $u_\ell Q_{k\ell'}$ is the k-th component of the rotated vector in the basis constituted by the vectors $\mathbf{i}_k$ and (b) in accordance with all that has been said in Subsection 1.4.5 the transpose of the tensor $\boldsymbol{Q}$ defined by the equation

$$\boldsymbol{Q} = \mathcal{Q}_{\ell k}\, \mathbf{i}_\ell \circ \mathbf{i}_k\,, \qquad\qquad \mathcal{Q}_{\ell k} = Q_{\ell' k} \tag{8.18}$$

is the rotation tensor $\boldsymbol{\mathcal{R}}$ that rotates the vector $\mathbf{u}$ into the vector $\overset{*}{\mathbf{u}}$.

A comparison of (8.16) and (8.18) shows that the tensor $\boldsymbol{\mathcal{R}}$ can be given in the following form:

$$\boldsymbol{\mathcal{R}} = \mathcal{Q}_{\ell' k}\, \mathbf{i}_k \circ \mathbf{i}_\ell = \mathbf{i}'_\ell \circ \mathbf{i}_\ell\,. \tag{8.19}$$

Remark 8.4: The expression rotation tensor has already been used for the tensor $\boldsymbol{R}$, which is included in the polar decomposition of the deformation gradient for describing the local rotations associated with the pure deformations. The use of the expression rotation tensor will, however, cause no misunderstanding in the sequel because its meaning will always turn out from the context.

Remark 8.5: In spite of equality (8.11), which is repeated here for completeness:

$$\underset{(3\times3)}{\underline{\boldsymbol{\mathcal{T}}}} = \underset{(3\times3)}{\underline{\mathbf{Q}}^T}\,,$$

the transformation tensor $\boldsymbol{\mathcal{T}}$ and the rotation tensor $\mathbf{Q}^T = \boldsymbol{\mathcal{R}}$ are different since the base tensors are not the same:

$$\boldsymbol{\mathcal{T}} = \mathcal{T}_{k\ell'}\, \mathbf{i}_k \circ \mathbf{i}'_\ell = Q_{\ell' k} \mathbf{i}_k \circ \mathbf{i}'_\ell \neq \boldsymbol{Q}^T = \mathcal{Q}_{\ell' k}\, \mathbf{i}_k \circ \mathbf{i}_\ell \tag{8.20}$$

It is worth emphasizing that the rotation tensor here is a known quantity since we also know the primed (rotated) coordinate system though the rotated vector is considered, naturally, in the unprimed coordinate system.

Remark 8.6: The tensor $\boldsymbol{Q}$ defined by equation (8.18) is proper orthogonal. Since $\boldsymbol{\mathcal{R}} = \mathbf{Q}^T$ the tensor $\boldsymbol{\mathcal{R}}$ is also proper orthogonal.

Remark 8.7: Let $\mathbf{t}$ be a vector and $\boldsymbol{W}$ be a tensor of order two. Then

$$\hat{\mathbf{t}} = \boldsymbol{Q} \cdot \mathbf{t} = (Q_{\ell' k}\, \mathbf{i}_\ell \circ \mathbf{i}_k) \cdot t_m \mathbf{i}_m = \underset{(1.54)_1}{\uparrow} = \\ = \mathbf{i}_\ell \underbrace{Q_{\ell' m} t_m}_{t'_\ell} = t'_\ell \mathbf{i}_\ell \neq t'_\ell \mathbf{i}'_\ell = \underset{(1.53)}{\uparrow} = \mathbf{t}' \tag{8.21a}$$

though

$$\underset{(3\times3)}{\underline{Q}}\ \underset{(3\times1)}{\underline{t}} = \underset{(3\times1)}{\underline{t}'} . \tag{8.21b}$$

These equations show the following: (a) the scalar components of the vector $\hat{\mathbf{t}}$ are the same as those of the vector $\mathbf{t}'$ but $\hat{\mathbf{t}} \neq \mathbf{t}'$ since $\mathbf{i}_\ell \neq \mathbf{i}'_\ell$; (b) mapping (8.21a) is a rotation.
We get in the same way:

$$\hat{\boldsymbol{W}} = \boldsymbol{Q} \cdot \boldsymbol{W} \cdot \boldsymbol{Q}^T = (Q_{k'm}\, \mathbf{i}_k \circ \mathbf{i}_m) \cdot (w_{rs}\, \mathbf{i}_r \circ \mathbf{i}_s) \cdot (Q_{\ell' n}\, \mathbf{i}_n \circ \mathbf{i}_\ell) = \underset{(1.73b)}{\uparrow} = \\ = \underbrace{Q_{k'm}\, Q_{\ell' n}\, w_{mn}}_{w'_{k\ell}}\, \mathbf{i}_k \circ \mathbf{i}_\ell = w'_{k\ell}\, \mathbf{i}_k \circ \mathbf{i}_\ell \neq w'_{k\ell}\, \mathbf{i}'_k \circ \mathbf{i}'_\ell = \underset{(1.73c)}{\uparrow} = \boldsymbol{W}' \tag{8.22a}$$

but

$$\underset{(3\times3)}{\underline{\mathbf{Q}}}\ \underset{(3\times3)}{\underline{\mathbf{W}}}\ \underset{(3\times3)}{\underline{\mathbf{Q}}^T} = \underset{(3\times3)}{\underline{\mathbf{W}'}} .$$

According to these two equations the scalar components (the matrices) of the tensors $\hat{\boldsymbol{W}}$ and $\boldsymbol{W}'$ are the same but $\hat{\boldsymbol{W}} \neq \boldsymbol{W}'$ since $\mathbf{i}_k \circ \mathbf{i}_\ell \neq \mathbf{i}'_k \circ \mathbf{i}'_\ell$.

Remark 8.8: Let us assume that the unprimed coordinate system is shifted by $\mathbf{x}_{OO'}$ and then it is rotated by $\boldsymbol{\mathcal{R}}$ in such a manner that the physical quantities in it move all together with the coordinate system resulting in the primed coordinate system with all the physical quantities being now in it – see Figure 8.2 for details. It is worthy of emphasizing the following:

(a) Let t be a scalar in the unprimed coordinate system. Then it holds that

$$\overset{*}{t} = t . \tag{8.23a}$$

(b) Let $\mathbf{t}$ be a vector in the unprimed coordinate system. The product

$$\overset{*}{\mathbf{t}} = \boldsymbol{\mathcal{R}} \cdot \mathbf{t} = (\mathbf{i}'_\ell \circ \mathbf{i}_\ell) \cdot t_m \mathbf{i}_m = t_\ell \mathbf{i}'_\ell \tag{8.23b}$$

yields the rotated vector, which is considered in the primed coordinate system. The scalar components of the vector $\mathbf{t}$ have not changed.

(c) Let $\boldsymbol{T}$ be a tensor in the unprimed coordinate system. The product

$$\overset{*}{\boldsymbol{T}} = \boldsymbol{\mathcal{R}} \cdot \boldsymbol{T} \cdot \boldsymbol{\mathcal{R}}^T = (\mathbf{i}'_m \circ \mathbf{i}_m) \cdot (t_{k\ell}\, \mathbf{i}_k \circ \mathbf{i}_\ell) \cdot (\mathbf{i}_n \circ \mathbf{i}'_n) = \\ = t_{mn} \mathbf{i}'_m \circ \mathbf{i}'_n \tag{8.23c}$$

yields the rotated tensor which is considered in the primed coordinate system. The scalar components of the tensor $\boldsymbol{T}$ have not changed.

8.2.2. The role of observers.

8.2.2.1. *Reference frames.* Section 1.4.6 in Chapter 1 clarifies the tensor concept by requiring that the components of a tensor quantity should satisfy certain transformation rules. In the present subsection we shall make the concept more precise by showing how the tensor concept is related to two physically important concepts namely to those of the objectivity and invariance of tensors. A reference frame is the model of a rigid body from where observations (time and distance measurements) are made by an observer. If we have two different reference frames then for the observer in the first reference frame

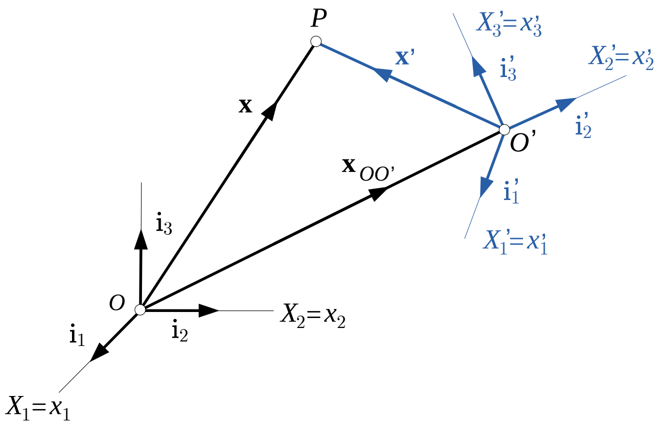

FIGURE 8.2. Two reference frames, the first is at rest the second is in motion

the second performs, in general, a rigid body motion with respect to the first one, and conversely for the observer in the second reference frame the first reference frame performs also a rigid body motion. In a reference frame we may have more than one coordinate system but one of these coordinate systems represents the reference frame itself, i.e., it replaces the rigid body with the observer on it. It is worth mentioning that we did not make a difference between reference frames and coordinate systems in Chapter 1 but spoke simply about coordinate systems only.

A tensor describes, in general, a physical quantity which, in most cases, is independent of the reference frame. The deformation measures are, for instance, reference frame independent quantities, or in other words they are independent of the observers. Though the tensor which describes a physical quantity is, in general, independent of the reference frame, however, the components of the tensor considered depend on what the reference frame is (what the coordinate system is we consider as reference frame).

Figure 8.2 shows two reference frames. The Cartesian coordinate system
$\{(x_1 = X_1, x_2 = X_2, x_3 = X_3)$ with unit vectors $\mathbf{i}_\ell\}$
$[(x'_1 = X'_1, x'_2 = X'_2, x'_3 = X'_3)$ with unit vectors $\mathbf{i}'_\ell]$

represent the {first} [second] reference frame. We shall assume that the unprimed reference frame is an inertial one and is at rest. We shall also assume that we have two observers. The first is located at the origin of the first reference frame (at the point O), the second at the origin of the second reference frame (at the point O'). The clocks the observers have are set to show the same point of time. In addition the observers are capable of measuring distances by using the same scale when performing measurements.

It is obvious from Figure 8.2 that

$$\mathbf{x} = \mathbf{x}_{OO'} + \mathbf{x}' \quad \text{or} \quad \mathbf{x}' = \mathbf{x} - \mathbf{x}_{OO'} \tag{8.24}$$

are the position vectors of the spatial point P in the two coordinate systems.

8.2.2.2. *Base vectors.* It can be seen from Figure 8.2 that the position vector of the material point P in the primed reference frame is

$$\underset{(x')}{\mathbf{x}'} = \underset{(x')}{\boldsymbol{\chi}}{}' = x'_k\,\mathbf{i}'_k \tag{8.25a}$$

for the second observer and

$$\underset{(x)}{\mathbf{x}'} = \underset{(x)}{\boldsymbol{\chi}}{}' = \underset{(x)}{x}{}'_\ell\,\mathbf{i}_\ell \tag{8.25b}$$

is for the first observer where the letters x and x' in parentheses identify the reference frame in which the corresponding quantities (vectors or vector components) are regarded (measured). It is obvious that in general

$$x'_k = \underset{(x')}{x}{}'_k \qquad \text{and} \qquad \underset{(x')}{x}{}'_k \neq \underset{(x)}{x}{}'_k\,.$$

The reference (or initial) configuration of the body (the region V° with boundary $A^\circ = \partial V^\circ$ the body occupies at time $t = t^\circ = 0$) is the same for both observers, however the two reference frames, which are, in general, different from each other for $t > 0$ since the first is at rest and the second is in motion with respect to the first one, may coincide with each other when the motion begins. The fact that they may coincide with each other for $t = 0$ does not violate generality. For this reason it is assumed in the sequel that the two reference frames coincide with each other at $t = t^\circ = 0$. Hence we make no difference in the notation of those quantities which belong to the initial state of the body.

Making use of $(8.24)_1$ we get

$$\mathbf{i}_\ell = \frac{\partial \mathbf{x}}{\partial x_\ell} = \frac{\partial \mathbf{x}}{\partial x'_k}\frac{\partial x'_k}{\partial x_\ell} = \Bigg(\underbrace{\frac{\partial \mathbf{x}_{OO'}}{\partial x'_k}}_{=\mathbf{0}} + \underbrace{\frac{\partial}{\partial x'_k}\underset{(x')}{\mathbf{x}'}}_{\mathbf{i}'_k}\Bigg)\frac{\partial x'_k}{\partial x_\ell} = \mathbf{i}'_k\frac{\partial x'_k}{\partial x_\ell}\,. \tag{8.26a}$$

Thus

$$\mathbf{i}'_m \cdot \mathbf{i}_\ell = \frac{\partial x'_m}{\partial x_\ell} \underset{(1.26b)}{=\uparrow=} Q_{m'k} \underset{(8.9)}{=\uparrow=} \mathcal{T}_{km'} \tag{8.26b}$$

which means that

$$\boxed{\mathbf{i}'_m = \frac{\partial x'_m}{\partial x_k}\,\mathbf{i}_k = Q_{m'k}\,\mathbf{i}_k = \mathbf{i}_k\,\mathcal{T}_{km'}\,, \qquad \mathbf{i}_\ell = \frac{\partial x'_n}{\partial x_\ell}\,\mathbf{i}'_n = Q_{kn'}\,\mathbf{i}'_n = \mathbf{i}'_n\mathcal{T}_{n'k}\,.} \tag{8.27}$$

Utilizing (8.24)$_2$ yields

$$\mathbf{i}'_k = \frac{\partial}{\partial x'_k}\,\mathbf{x}'_{(x')} = \frac{\partial}{\partial x_\ell}\,\mathbf{x}'_{(x')}\frac{\partial x_\ell}{\partial x'_k} = \left(\frac{\partial \mathbf{x}}{\partial x_\ell} - \underbrace{\frac{\partial \mathbf{x}_{OO'}}{\partial x_\ell}}_{=\mathbf{0}}\right)\frac{\partial x_\ell}{\partial x'_k} = \mathbf{i}_\ell\frac{\partial x_\ell}{\partial x'_k}. \tag{8.28a}$$

Hence,

$$\mathbf{i}_n \cdot \mathbf{i}'_k = \frac{\partial x_n}{\partial x'_k} \underset{(1.29b)}{=\uparrow=} Q_{nk'} \underset{(8.9)}{=\uparrow=} \mathcal{T}_{k'n} \tag{8.28b}$$

which means that

$$\boxed{\mathbf{i}_n = \frac{\partial x_n}{\partial x'_k}\,\mathbf{i}'_k = Q_{nk'}\,\mathbf{i}'_k = \mathbf{i}'_k\,\mathcal{T}_{k'n}\,, \quad \mathbf{i}'_k = \frac{\partial x_n}{\partial x'_k}\mathbf{i}_n = Q_{k'n}\,\mathbf{i}_n = \mathbf{i}_n\,\mathcal{T}_{nk'}\,.} \tag{8.29}$$

8.2.2.3. *Equivalence of nabla operators.* It follows from the manipulation

$$\nabla = \frac{\partial}{\partial x_k}\,\mathbf{i}_k = \frac{\partial}{\partial x_{m'}}\frac{\partial x_{m'}}{\partial x_k}\,\mathbf{i}_k \underset{(8.27)}{=\uparrow=} \frac{\partial}{\partial x_{m'}}\mathbf{i}_{m'} = \nabla' \tag{8.30a}$$

that the operator nabla is a reference frame independent quantity. This statement is valid for any point of time. Hence

$$\nabla^\circ = \nabla^{\circ\prime} \tag{8.30b}$$

at $t = t^\circ = 0$ independently of the fact whether the primed and unprimed reference frames coincide with each other or not when the motion begins.

8.2.3. Objective tensors.

8.2.3.1. *Definitions.* Let us denote a tensor of order n by a superscript n in a pair of parentheses preceding the letter that identifies the tensor in question. For example ${}^{(n)}\boldsymbol{T}'$ $(n = 0, 1, 2, 3, ...)$ is a tensor of order n in the moving reference frame. Accordingly

$$\begin{array}{rll} {}^{(0)}\boldsymbol{T}' = t' & & \text{is a scalar;} \\ {}^{(1)}\boldsymbol{T}' = \mathbf{t} = t'_\ell \mathbf{i}'_\ell & & \text{is a vector;} \\ {}^{(2)}\boldsymbol{T}' = \boldsymbol{T} = t'_{k\ell}\,\mathbf{i}'_k \circ \mathbf{i}'_\ell & & \text{is a tensor;} \\ {}^{(3)}\boldsymbol{T}' = t'_{k\ell r}\,\mathbf{i}'_k \circ \mathbf{i}'_\ell \circ \mathbf{i}'_r & & \text{is a triad;} \\ {}^{(4)}\boldsymbol{T}' = t'_{k\ell rs}\,\mathbf{i}'_k \circ \mathbf{i}'_\ell \circ \mathbf{i}'_r \circ \mathbf{i}'_s & & \text{is a tetrad;} \\ \text{etc.} & & \end{array}$$

It should be emphasized that the second observer is, naturally, capable of recording these tensors in the moving reference frame.

Note that the above notation convention is in full accordance with the notation convention introduced in Subsection 1.4.6.2 for tensors of order higher than two.

We shall now introduce the concept of the Rayleigh product [85]. The definition is as follows:

$$\boldsymbol{\mathcal{T}} * {}^{(0)}\boldsymbol{T}' = t', \tag{8.31a}$$

$$\boldsymbol{\mathcal{T}} * {}^{(n)}\boldsymbol{T}' = \boldsymbol{\mathcal{T}} * \left(t'_{k\ell\ldots p}\mathbf{i}'_k \circ \mathbf{i}'_\ell \circ \cdots \circ \mathbf{i}'_p\right) = \\ = t'_{k\ell\ldots p}\left(\boldsymbol{\mathcal{T}} \cdot \mathbf{i}'_k\right) \circ \left(\boldsymbol{\mathcal{T}} \cdot \mathbf{i}'_\ell\right) \circ \cdots \circ \left(\boldsymbol{\mathcal{T}} \cdot \mathbf{i}'_p\right), \quad n \geq 1 \tag{8.31b}$$

which shows that the Rayleigh product has no effect on the scalars: for $n = 0$ it coincides with the identity.

REMARK 8.9: The above definition can easily be generalized since, for instance, the first factor can be any tensor of order two – for our aims, however, the above definition is sufficient.

Let us assume that a tensor $\boldsymbol{T}$ of arbitrary order describes the same physical phenomenon in the unprimed and primed coordinate systems. We shall call it objective under the change of observer if

$$ {}^{(n)}\boldsymbol{T} = \boldsymbol{\mathcal{T}} * {}^{(n)}\boldsymbol{T}' . \tag{8.32}$$

Accordingly a scalar is objective if

$$ t = t' ; \tag{8.33a}$$

a vector is objective if

$$ \mathbf{t} = \boldsymbol{\mathcal{T}} * \mathbf{t}' = \boldsymbol{\mathcal{T}} \cdot \mathbf{t}' , \qquad t_m = \mathcal{T}_{m\ell'}\, t'_\ell ; \tag{8.33b}$$

a tensor is objective if

$$ \boldsymbol{T} = \boldsymbol{\mathcal{T}} * \boldsymbol{T}' = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{T}' \cdot \boldsymbol{\mathcal{T}}^T , \qquad t_{mn} = \mathcal{T}_{mk'}\, t'_{k\ell}\, \mathcal{T}_{\ell' n} = \mathcal{T}_{mk'}\, \mathcal{T}_{n\ell'}\, t'_{k\ell} ; \tag{8.33c}$$

a triad is objective if

$$ {}^{(3)}\boldsymbol{T} = \boldsymbol{\mathcal{T}} * {}^{(3)}\boldsymbol{T}' , \qquad t_{mnp} = \mathcal{T}_{mk'}\, \mathcal{T}_{n\ell'}\, \mathcal{T}_{pr'}\, t'_{k\ell r} ; \tag{8.33d}$$

a tetrad is objective if

$$ {}^{(4)}\boldsymbol{T} = \boldsymbol{\mathcal{T}} * {}^{(4)}\boldsymbol{T}' , \qquad t_{mnpq} = \mathcal{T}_{mk'}\, \mathcal{T}_{n\ell'}\, \mathcal{T}_{pr'}\, \mathcal{T}_{qs'}\, t'_{k\ell rs} . \tag{8.33e}$$

REMARK 8.10: Note that the component form of equations (8.33) coincide with the relations that are presented in the forth column of Table 1 since it follows from equation (8.11) that $\mathcal{T}_{m\ell'} = Q_{\ell' m}$.

REMARK 8.11: Assume that the vector $\mathbf{n}$ is objective, i.e., it holds that $\mathbf{n} = \boldsymbol{\mathcal{T}} \cdot \mathbf{n}'$. Assume further that the tensor $\boldsymbol{T}$ is also objective. Then it holds

$$ \boldsymbol{T} \cdot \mathbf{n} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{T}' \cdot \underbrace{\boldsymbol{\mathcal{T}}^T \cdot \boldsymbol{\mathcal{T}}}_{\boldsymbol{1}} \cdot \mathbf{n} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{T}' \cdot \mathbf{n} . \tag{8.34}$$

This result means that the dot product of an objective tensor and an objective vector is also an objective vector.

REMARK 8.12: The above definition of objectivity is based on that of Ogden [68, 80]. Note that other definitions are also possible – see for instance [85].

A tensor of arbitrary order is called invariant if it holds that

$$ {}^{(n)}\boldsymbol{T} = \boldsymbol{\mathcal{T}} * {}^{(n)}\boldsymbol{T}' = {}^{(n)}\boldsymbol{T}' . \tag{8.35}$$

8.2.3.2. *Objectivity of some deformation characteristics.* Consider first the deformation gradients. The manipulation

$$\boldsymbol{F} = \mathbf{x} \circ \nabla^\circ = \big(\mathbf{x}_{OO'} + \underset{(x)}{\mathbf{x}'}\big) \circ \nabla^\circ = \underbrace{\mathbf{x}_{OO'} \circ \nabla^\circ}_{=\boldsymbol{O}} + \underset{(x)}{\mathbf{x}'} \circ \nabla^\circ =$$

$$= \underset{\underset{(x)}{\mathbf{x}'} = \boldsymbol{\mathcal{T}} \cdot \underset{(x')}{\mathbf{x}'}}{\uparrow} \quad \underset{\nabla^\circ = \nabla^{\circ\prime}}{\uparrow} = \boldsymbol{\mathcal{T}} \cdot \big(\underset{(x')}{\mathbf{x}'} \circ \nabla^{\circ\prime}\big) = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{F}', \tag{8.36a}$$

in which it is taken into account that $\boldsymbol{\mathcal{T}} \cdot \nabla^{\circ\prime} = \mathbf{0}$, shows that deformation gradient behaves like a vector.

Making use of (8.36a) and the definition of $\boldsymbol{U}$ it can be checked that the right Cauchy-Green tensor and the right stretch tensor satisfies the following relations:

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{F}'^T \cdot \boldsymbol{\mathcal{T}}^T \cdot \boldsymbol{\mathcal{T}} \cdot \boldsymbol{F}' = \boldsymbol{F}'^T \cdot \boldsymbol{F}' = \boldsymbol{C}' \tag{8.36b}$$

and

$$\boldsymbol{U} = \sqrt{\boldsymbol{C}} = \boldsymbol{U}', \qquad \boldsymbol{U}^{-1} = \boldsymbol{U}'^{-1}. \tag{8.36c}$$

Utilizing the polar decomposition theorem and the previous two formulae we get

$$\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{F}' \cdot \boldsymbol{U}'^{-1} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{R}' \tag{8.36d}$$

for the rotation tensor,

$$\boldsymbol{v} = \boldsymbol{F} \cdot \boldsymbol{R}^T = \boldsymbol{F} \cdot \boldsymbol{R}'^T \cdot \boldsymbol{\mathcal{T}}^T = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{F}' \cdot \boldsymbol{R}'^T \cdot \boldsymbol{\mathcal{T}}^T = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{v}' \cdot \boldsymbol{\mathcal{T}}^T \tag{8.36e}$$

for the left stretch tensor,

$$\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{F}' \cdot \boldsymbol{F}'^T \cdot \boldsymbol{\mathcal{T}}^T = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{b}' \cdot \boldsymbol{\mathcal{T}}^T \tag{8.36f}$$

for the Cauchy strain tensor,

$$\boldsymbol{b}^{-1} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{b}'^{-1} \cdot \boldsymbol{\mathcal{T}}^T \tag{8.36g}$$

for the left Cauchy-Green tensor,

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{1}) = \frac{1}{2}\big(\boldsymbol{C}' - \boldsymbol{1}\big) = \boldsymbol{E}' \tag{8.36h}$$

for the Green-Lagrange strain tensor and

$$\boldsymbol{e} = \frac{1}{2}\big(\boldsymbol{1} - \boldsymbol{b}^{-1}\big) = \frac{1}{2}\Big((\boldsymbol{1} - \boldsymbol{\mathcal{T}} \cdot \boldsymbol{b}'^{-1} \cdot \boldsymbol{\mathcal{T}}^T\Big) =$$

$$= \boldsymbol{\mathcal{T}} \cdot \frac{1}{2}\big(\boldsymbol{1} - \boldsymbol{b}'^{-1}\big) \cdot \boldsymbol{\mathcal{T}}^T = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{e}' \cdot \boldsymbol{\mathcal{T}}^T \tag{8.36i}$$

for the Euler-Almansi strain tensor. It is also obvious that

$$J = \det(\boldsymbol{F}) = \det(\boldsymbol{\mathcal{T}} \cdot \boldsymbol{F}') = \underbrace{\det(\boldsymbol{\mathcal{T}})}_{=1} \det(\boldsymbol{F}') = \det(\boldsymbol{F}') = J'. \tag{8.37}$$

Consequently, it follows from (2.94) that the volume elements measured by the two observers are the same.

It also holds that

$$\mathbf{n}\cdot\boldsymbol{b}\cdot\mathbf{n} = (\boldsymbol{\mathcal{T}}\cdot\mathbf{n}')\cdot\boldsymbol{\mathcal{T}}\cdot\boldsymbol{b}'\cdot\boldsymbol{\mathcal{T}}^T\cdot(\boldsymbol{\mathcal{T}}\cdot\mathbf{n}') = \\ = \mathbf{n}'\cdot\underbrace{\boldsymbol{\mathcal{T}}^T\cdot\boldsymbol{\mathcal{T}}}_{=\boldsymbol{1}}\cdot\boldsymbol{b}'\cdot\underbrace{\boldsymbol{\mathcal{T}}^T\cdot\boldsymbol{\mathcal{T}}}_{=\boldsymbol{1}}\cdot\mathbf{n}' = \mathbf{n}'\cdot\boldsymbol{b}'\cdot\mathbf{n}'\,.$$

Hence, it follows from formula $(2.91)_2$ devised for the scalar surface elements that

$$\mathrm{d}A = \mathrm{d}A'\,. \tag{8.38}$$

Conclusions concerning the objectivity of the considered deformation characteristics may now be drown:

- The deformation gradient $\boldsymbol{F}$ and the rotation tensor $\boldsymbol{R}$ are two two-point tensors. They behave as if they were vectors.
- The left stretch tensor $\boldsymbol{v}$, the Cauchy strain tensor $\boldsymbol{b}$, the left Cauchy-Green tensor $\boldsymbol{b}^{-1}$ and the Euler-Almansi strain tensor $\boldsymbol{e}$ are objective.
- The right Cauchy-Green tensor $\boldsymbol{C}$, the right stretch tensor $\boldsymbol{U}$ and the Green-Lagrange strain tensor $\boldsymbol{E}$ are invariant.
- The volume element $\mathrm{d}V$ and the scalar surface element $\mathrm{d}A$ are also invariant. This conclusion is consistent with the intuitive notion that the local volume and surface changes are independent of the kinematical description (of the observers).

8.2.3.3. *Velocity and acceleration.* For the second observer the following time derivative is the velocity of the material point at P – see Figure 8.2:

$$\mathbf{v}' = \big(\underset{(x')}{\mathbf{x}'}\big)^{\bullet}\,. \tag{8.39}$$

For the first observer, however,

$$\mathbf{v} = (\mathbf{x})^{\bullet} = \Big(\mathbf{x}_{OO'} + \boldsymbol{\mathcal{T}}\cdot\underset{(x)}{\mathbf{x}'}\Big)^{\bullet} = \big(\mathbf{x}_{OO'}+\boldsymbol{\mathcal{T}}\cdot\underset{(x')}{\mathbf{x}'}\big)^{\bullet} = (\mathbf{x}_{OO'})^{\bullet}+(\boldsymbol{\mathcal{T}})^{\bullet}\cdot\underset{(x')}{\mathbf{x}'}+\boldsymbol{\mathcal{T}}\cdot\big(\underset{(x')}{\mathbf{x}'}\big)^{\bullet}$$

in which $\underset{(x')}{\mathbf{x}'} = \boldsymbol{\mathcal{T}}^T\cdot\underset{(x)}{\mathbf{x}'}$ since the first observer can record $\underset{(x)}{\mathbf{x}'}$ only.

Recalling now equation (8.15a), which shows that the product $(\boldsymbol{\mathcal{T}})^{\bullet}\cdot\boldsymbol{\mathcal{T}}^T$ is skew, we get

$$\mathbf{v} = (\mathbf{x}_{OO'})^{\bullet} + \underbrace{(\boldsymbol{\mathcal{T}})^{\bullet}\cdot\boldsymbol{\mathcal{T}}^T}_{=\boldsymbol{\Omega}}\cdot\underset{(x)}{\mathbf{x}'} + \boldsymbol{\mathcal{T}}\cdot\big(\underset{(x')}{\mathbf{x}'}\big)^{\bullet}$$

from where substituting the cross product $\hat{\boldsymbol{\omega}}\times\underset{(x)}{\mathbf{x}'}$ for the dot product $\boldsymbol{\Omega}\cdot\underset{(x)}{\mathbf{x}'}$ yields

$$\mathbf{v} = (\mathbf{x}_{OO'})^{\bullet} + \hat{\boldsymbol{\omega}}\times\underset{(x)}{\mathbf{x}'} + \boldsymbol{\mathcal{T}}\cdot\big(\underset{(x')}{\mathbf{x}'}\big)^{\bullet}. \tag{8.40}$$

Here

$$\boldsymbol{\mathcal{T}}\cdot\big(\underset{(x')}{\mathbf{x}'}\big)^{\bullet}$$

is the relative velocity of the material point at P, i.e., the velocity of the material point with respect to the moving reference frame in the form the first observer can see it,

$$(\mathbf{x}_{OO'})^{\cdot} + \hat{\boldsymbol{\omega}} \times \underset{(x)}{\mathbf{x}'}$$

is the transport velocity, i.e., the velocity at that point of the moving reference frame which coincides with the material point considered at the instant of the investigation,

$$(\mathbf{x}_{OO'})^{\cdot}$$

is the velocity of the point O' with respect to the first reference frame and finally

$$\hat{\boldsymbol{\omega}}$$

is the relative angular velocity – in fact the angular velocity of the moving reference frame with respect to the first reference frame.

For the second observer

$$\mathbf{a}' = \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot\cdot} \tag{8.41}$$

is the acceleration.

For the first observer we have

$$\begin{aligned}
\mathbf{a} = (\mathbf{v})^{\cdot} &= (\mathbf{x}_{OO'})^{\cdot\cdot} + (\hat{\boldsymbol{\omega}})^{\cdot} \times \underset{(x)}{\mathbf{x}'} + \hat{\boldsymbol{\omega}} \times \Big(\underset{(x)}{\mathbf{x}'}\Big)^{\cdot} + (\boldsymbol{\mathcal{T}})^{\cdot} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot} + \boldsymbol{\mathcal{T}} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot\cdot} = \\
&= (\mathbf{x}_{OO'})^{\cdot\cdot} + (\hat{\boldsymbol{\omega}})^{\cdot} \times \underset{(x)}{\mathbf{x}'} + \hat{\boldsymbol{\omega}} \times \Big(\boldsymbol{\mathcal{T}} \cdot \underset{(x')}{\mathbf{x}'}\Big)^{\cdot} + (\boldsymbol{\mathcal{T}})^{\cdot} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot} + \boldsymbol{\mathcal{T}} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot\cdot} = \\
&= (\mathbf{x}_{OO'})^{\cdot\cdot} + (\hat{\boldsymbol{\omega}})^{\cdot} \times \underset{(x)}{\mathbf{x}'} + \hat{\boldsymbol{\omega}} \times \Big((\boldsymbol{\mathcal{T}})^{\cdot} \cdot \boldsymbol{\mathcal{T}}^T \cdot \boldsymbol{\mathcal{T}} \cdot \underset{(x')}{\mathbf{x}'}\Big) + \\
&\qquad + 2(\boldsymbol{\mathcal{T}})^{\cdot} \cdot \boldsymbol{\mathcal{T}}^T \cdot \boldsymbol{\mathcal{T}} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot} + \boldsymbol{\mathcal{T}} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot\cdot}.
\end{aligned}$$

Hence

$$\mathbf{a} = (\mathbf{v})^{\cdot} = (\mathbf{x}_{OO'})^{\cdot\cdot} + (\hat{\boldsymbol{\omega}})^{\cdot} \times \underset{(x)}{\mathbf{x}'} + \hat{\boldsymbol{\omega}} \times \Big(\hat{\boldsymbol{\omega}} \times \boldsymbol{\mathcal{T}} \cdot \underset{(x')}{\mathbf{x}'}\Big) + 2\hat{\boldsymbol{\omega}} \times \boldsymbol{\mathcal{T}} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot} + \boldsymbol{\mathcal{T}} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot\cdot}. \tag{8.42}$$

in which

$$\boldsymbol{\mathcal{T}} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot\cdot}$$

is the relative acceleration of the material point P, i.e., the acceleration of the material point with respect to the moving reference frame in the form the first observer can record it,

$$2\hat{\boldsymbol{\omega}} \times \boldsymbol{\mathcal{T}} \cdot \Big(\underset{(x')}{\mathbf{x}'}\Big)^{\cdot}$$

is the Coriolis[1] acceleration [7],

$$(\mathbf{x}_{OO'})^{\cdot\cdot} + (\hat{\boldsymbol{\omega}})^{\cdot} \times \underset{(x)}{\mathbf{x}'} + \hat{\boldsymbol{\omega}} \times \Big(\hat{\boldsymbol{\omega}} \times \boldsymbol{\mathcal{T}} \cdot \underset{(x')}{\mathbf{x}'}\Big)$$

is the transport acceleration, i.e., the acceleration of that point of the moving reference frame which coincides with the material point at P at the instant of the investigation,

[1] Gaspard Gustave de Coriolis, 1792-1843

$(\mathbf{x}_{OO'})^{\cdot\cdot}$
is the acceleration of the point O' with respect to the first reference frame,

$\hat{\boldsymbol{\omega}}^{\cdot}$
is the relative angular acceleration – in fact the angular acceleration of the moving reference frame with respect to the first reference frame and

$\hat{\boldsymbol{\omega}} \times \left(\hat{\boldsymbol{\omega}} \times \boldsymbol{\mathcal{T}} \cdot \underset{(x')}{\mathbf{x}'}\right)$
is the centripetal acceleration.

8.2.3.4. *Strain rate tensor and its additive resolution.* As regards the velocity gradient we have

$$\boldsymbol{\ell} = \underbrace{\boldsymbol{\ell} \cdot \boldsymbol{F}}_{=(\boldsymbol{F})^{\cdot}} \cdot \boldsymbol{F}^{-1} = (\boldsymbol{F})^{\cdot} \cdot \boldsymbol{F}^{-1} = \underset{(8.36a)}{\uparrow} =$$

$$= (\boldsymbol{\mathcal{T}} \cdot \boldsymbol{F}')^{\cdot} \cdot \left(\boldsymbol{F}'^{-1} \cdot \boldsymbol{\mathcal{T}}^T\right) = (\boldsymbol{\mathcal{T}})^{\cdot} \cdot \boldsymbol{F}' \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{\mathcal{T}}^T + \boldsymbol{\mathcal{T}} \cdot \underbrace{(\boldsymbol{F}')^{\cdot}}_{\boldsymbol{\ell}' \cdot \boldsymbol{F}'} \cdot \boldsymbol{F}'^{-1} \cdot \boldsymbol{\mathcal{T}}^T .$$

Hence

$$\boldsymbol{\ell} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{\ell}' \cdot \boldsymbol{\mathcal{T}}^T + (\boldsymbol{\mathcal{T}})^{\cdot} \cdot \boldsymbol{\mathcal{T}}^T , \tag{8.43}$$

where $\boldsymbol{\ell} = \boldsymbol{d} + \boldsymbol{\Omega}$, $\boldsymbol{\ell}' = \boldsymbol{d}' + \boldsymbol{\Omega}'$, and $(\boldsymbol{\mathcal{T}})^{\cdot} \cdot \boldsymbol{\mathcal{T}}^T$ is skew. Consequently,

$$\boldsymbol{d} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{d}' \cdot \boldsymbol{\mathcal{T}}^T \tag{8.44a}$$

and

$$\boldsymbol{\Omega} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{\Omega}' \cdot \boldsymbol{\mathcal{T}}^T + (\boldsymbol{\mathcal{T}})^{\cdot} \cdot \boldsymbol{\mathcal{T}}^T . \tag{8.44b}$$

Equation (8.44a) shows that the strain rate tensor $\boldsymbol{d}$ is objective. The spin tensor is, however, not objective due to the presence of the term $(\boldsymbol{\mathcal{T}})^{\cdot} \cdot \boldsymbol{\mathcal{T}}^T$ on the right side of equation (8.44b).

8.3. Fundamental principles for the constitutive equations

8.3.1. What equations are missing. It was clarified in Subsection 8.1 that the equations established so far should be supplemented by further equations since the number of equations we have established so far is less than that of unknowns. As regards the isothermal case it is an open issue how the stresses are related to the deformations. These equations are the material (or constitutive) equations which should provide one-to-one relationships between the various stress and strain measures.

The kinematic equations have been established by using geometrical considerations and are, therefore, exact, i.e., no approximations were applied when deriving them within the framework of the non-linear deformation theory. As regards the equations of motion those are based on the equivalence of the external and effective forces and in this sense they are also exact. The constitutive equations (the stress strain relations) should be based partly on experimental investigations partly on theoretical considerations since we can measure deformations only. The stresses themselves are not measurable quantities. Consequently, the constitutive equations are not exact but have an approximative character. On the other hand their mathematical form should satisfy some fundamental requirements known as principles of the material theory .

8.3.2. Fundamental principles. The fundamental principles the constitutive relations should meet are related both to fluids and to solid bodies. However, special emphasis is placed on the solid mechanical behavior and the effects other than mechanical (electrical for instance) are left out of consideration.

Principle of determinism of stress [47, 74]: *The stress at a material point of the body is determined by the history of motion of that body.*

If needed the past motion (or the history of motion) in the finite time interval $[0, t]$ will be denoted by

$$\boldsymbol{\chi}(\mathbf{X};\tau)|_{\tau=0}^{t} \,. \tag{8.45}$$

It is an experimental observation that the stresses at a material point within the body are independent of the motion of any other point within the body provided that the distance between these two points is greater than a given limit. If this limit is finite we speak about non-local theories. If this limit is infinitesimal, i.e., if the neighborhood of influence the other points have concerning the stresses at the point considered is as small as possible we speak about local action and the material of the body is called simple material. In this book we confine ourselves to considering simple materials only for which it holds the

Principle of local action [74, 79]: *In determining the stress in a given material point P the motion outside an arbitrarily small neighborhood of P can be disregarded.*

It is a fundamental issue how the introduced strain and stress measures depend on the observers. As regards the quantities of kinematic nature Subsection 8.2.3 clarifies the observer-dependence of these quantities. For the stress measures, however, a further requirement is needed:

Principle of material objectivity (or principle of material frame-indifference) [74, 82, 85]: *The material behavior must be independent of the observers, i.e., the constitutive equations should be invariant if the reference frame changes.*

8.4. Cauchy elastic bodies

8.4.1. The effect of local action. Assume that the body considered is stress free in the initial configuration – we assume that it coincides with the reference configuration. A body is said to be elastic if after being deformed it returns to its original shape and size when the forces causing the deformation are removed. We can also say that the body with this property behaves elastically.

In Subsection 8.4.2 we restrict our attention to the isothermal material behavior only, i.e, it is assumed that the absolute temperature is constant within the body.

Let Q° and P° be two different material points within the body. The displacement the material point Q° has with respect to that of the material point P° is given by the equation

$$x_\ell(Q^\circ) - x_\ell(P^\circ) = \chi_\ell[\boldsymbol{X}(Q^\circ);t] - \chi_\ell[\boldsymbol{X}(P^\circ);t] \tag{8.46}$$

in which differentiability is assumed for the motion law $\boldsymbol{\chi}(\boldsymbol{X};t)$. If Q° tends to P° in accordance with (2.13) we have

$$x_\ell(Q^\circ) - x_\ell(P^\circ) = \Delta x_\ell = \frac{\partial \chi_\ell}{\partial X_A}\mathrm{d}X_A = F_{\ell A}\mathrm{d}X_A\,. \tag{8.47}$$

This equation shows that the deformation state in an infinitesimal neighborhood of the point P° in the body is determined by the deformation gradient $F_{\ell A}$. Recalling the principle of local action we conclude that deformation state should uniquely determine the stress state in this infinitesimal environment. Hence the Cauchy stress tensor is a function of the deformation gradient at P°: $\boldsymbol{t} = \mathfrak{r}^t(\boldsymbol{F})$. Since the material properties may change within the body

$$\boldsymbol{t} = \mathfrak{r}^t(\boldsymbol{F}(\mathbf{X};t),\mathbf{X}) \tag{8.48}$$

is the general form of the previous equation. Here the function $\mathfrak{r}^t(\boldsymbol{F}(\mathbf{X};t),\mathbf{X})$ is called response function [74].

The body for which equation (8.48) is the constitutive equation is called Cauchy elastic body.

The body is said to be homogeneous if the material properties are the same for each particle within the body (everywhere within the body). For homogeneous bodies constitutive equation (8.48), which might also be called as stress relation, is simplified to the form

$$\boldsymbol{t} = \mathfrak{r}^t(\boldsymbol{F}(\mathbf{X};t))\,. \tag{8.49}$$

In what follows it is assumed that the body considered is homogeneous.

8.4.2. Independence of the material behavior from the observers. It is obvious that the independence of the material behavior from the observers should result in further restrictions on the response function. Since the material behavior should be the same for the observers in the unprimed and primed coordinate systems the mathematical form of the response function have to be the same for the two observers. With (8.36a) and (8.33c) we have

$$\boldsymbol{t} = \mathfrak{r}^t(\boldsymbol{F}) = \mathfrak{r}^t(\boldsymbol{\mathcal{T}}\cdot\boldsymbol{F}')\,,\quad \boldsymbol{t} = \boldsymbol{\mathcal{T}}\cdot\boldsymbol{t}'\cdot\boldsymbol{\mathcal{T}}^T = \boldsymbol{\mathcal{T}}\cdot\mathfrak{r}^t\cdot\boldsymbol{\mathcal{T}}^T = \boldsymbol{\mathcal{T}}\cdot\mathfrak{r}^t(\boldsymbol{F}')\cdot\boldsymbol{\mathcal{T}}^T, \tag{8.50}$$

from where it follows that

$$\mathfrak{r}^t(\boldsymbol{F}) = \boldsymbol{\mathcal{T}}\cdot\mathfrak{r}^t(\boldsymbol{F}')\cdot\boldsymbol{\mathcal{T}}^T\,. \tag{8.51}$$

If we utilize the polar decomposition theorem we may rewrite the previous equation:

$$\mathfrak{r}^t(\boldsymbol{F}) = \underset{\boldsymbol{F}'=\boldsymbol{\mathcal{T}}^T\cdot\boldsymbol{F}}{\uparrow} = \boldsymbol{\mathcal{T}}\cdot\mathfrak{r}^t(\boldsymbol{\mathcal{T}}^T\cdot\boldsymbol{F})\cdot\boldsymbol{\mathcal{T}}^T = \underset{\boldsymbol{F}=\boldsymbol{R}\cdot\boldsymbol{U}}{\uparrow} = \\ = \boldsymbol{\mathcal{T}}\cdot\mathfrak{r}^t(\boldsymbol{\mathcal{T}}^T\cdot\boldsymbol{R}\cdot\boldsymbol{U})\cdot\boldsymbol{\mathcal{T}}^T \tag{8.52}$$

This equation should be satisfied for any transformation tensor $\boldsymbol{\mathcal{T}}$, i.e., for any proper orthogonal tensor. Consequently, if we write $\boldsymbol{R}$ for $\boldsymbol{\mathcal{T}}$ and take into account that $\boldsymbol{R}^T \cdot \boldsymbol{R} = \boldsymbol{1}$ we get

$$\boldsymbol{t} = \mathfrak{r}^t(\boldsymbol{F}) = \boldsymbol{R} \cdot \mathfrak{r}^t(\boldsymbol{U}) \cdot \boldsymbol{R}^T . \tag{8.53}$$

This equation should hold for any proper orthogonal tensor $\boldsymbol{R}$. Consequently, if we write $\boldsymbol{\mathcal{T}}$ for $\boldsymbol{R}$ interchanging again the role of these two tensors and utilize the fact that $\boldsymbol{U} = \boldsymbol{U}'$ we have:

$$\boldsymbol{t} = \mathfrak{r}^t(\boldsymbol{F}) = \boldsymbol{\mathcal{T}} \cdot \mathfrak{r}^t(\boldsymbol{U}') \cdot \boldsymbol{\mathcal{T}}^T , \tag{8.54}$$

which shows that the response function $\mathfrak{r}^t$ is objective if it is a function of the right stretch tensor $\boldsymbol{U}$.

Recalling equation (5.27) for the second Piola-Kirchoff stress tensor yields

$$\boldsymbol{S} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot \boldsymbol{F}^{-T} = \underset{J=\det(\boldsymbol{U})}{\uparrow} \quad \underset{\boldsymbol{F}^{-1}=\boldsymbol{U}^{-1}\cdot\boldsymbol{R}^T}{\uparrow} = \\ = \det(\boldsymbol{U})\, \boldsymbol{U}^{-1} \cdot \boldsymbol{R}^T \cdot \boldsymbol{t} \cdot \boldsymbol{R} \cdot \boldsymbol{U}^{-1} \tag{8.55}$$

from where substituting (8.53) we obtain

$$\boldsymbol{S} = \det(\boldsymbol{U})\, \boldsymbol{U}^{-1} \cdot \boldsymbol{R}^T \cdot \boldsymbol{R} \cdot \mathfrak{r}^t(\boldsymbol{U}) \cdot \boldsymbol{R}^T \cdot \boldsymbol{R} \cdot \boldsymbol{U}^{-1} = \\ = \underset{\boldsymbol{R}^T\cdot\boldsymbol{R}=\boldsymbol{1}}{\uparrow} = \underbrace{\det(\boldsymbol{U})\, \boldsymbol{U}^{-1} \cdot \mathfrak{r}^t(\boldsymbol{U}) \cdot \boldsymbol{U}^{-1}}_{\mathfrak{r}^S(\boldsymbol{U})} ,$$

where $\mathfrak{r}^S(\boldsymbol{U})$ is the response function that belongs to the second Piola-Kirchoff stress tensor. Since $\boldsymbol{U} = \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{1} + 2\boldsymbol{E}}$ we may assume that the response function $\mathfrak{r}^S$ is a function of the Green-Lagrange strain tensor: $\mathfrak{r}^S = \mathfrak{r}^S(\boldsymbol{E})$. Since $\boldsymbol{S} = \boldsymbol{S}'$ and $\boldsymbol{E} = \boldsymbol{E}'$ it follows that the material behavior described by the response function $\mathfrak{r}^S = \mathfrak{r}^S(\boldsymbol{E})$ is independent of the observers.

8.4.3. Isotropic material. Let us assume that the Cauchy stress is a function of the Cauchy strain tensor $\boldsymbol{b}$ defined by equation (2.49). Then

$$\boldsymbol{t} = \mathfrak{r}^b(\boldsymbol{b}) , \tag{8.56}$$

where $\mathfrak{r}^b$ is the response function for which it is expected that the objectivity requirements be fulfilled. For the second observer $\boldsymbol{t}' = \mathfrak{r}^b(\boldsymbol{b}')$ is the stress tensor since the response function $= \mathfrak{r}^b$ should naturally be the same for the two observers. Hence

$$\boldsymbol{t} = \mathfrak{r}^b(\boldsymbol{b}) = \mathfrak{r}^b(\boldsymbol{\mathcal{T}} \cdot \boldsymbol{b}' \cdot \boldsymbol{\mathcal{T}}^T) \tag{8.57a}$$

and

$$\boldsymbol{t} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{t}' \cdot \boldsymbol{\mathcal{T}}^T = \boldsymbol{\mathcal{T}} \cdot \mathfrak{r}^b(\boldsymbol{b}') \cdot \boldsymbol{\mathcal{T}}^T . \tag{8.57b}$$

Comparing (8.57a) and (8.57b) yields

$$\boldsymbol{\mathcal{T}} \cdot \mathfrak{r}^b(\boldsymbol{b}') \cdot \boldsymbol{\mathcal{T}}^T = \mathfrak{r}^b(\boldsymbol{\mathcal{T}} \cdot \boldsymbol{b}' \cdot \boldsymbol{\mathcal{T}}^T) = \mathfrak{r}^b(\boldsymbol{b}) = \boldsymbol{t} . \tag{8.58}$$

If the response function $\mathfrak{r}^b$ satisfies this equation then the objectivity requirements (the requirement that the material behavior shoul be independent of the observers) is fulfilled.

In the sequel it is assumed that the response function $\mathfrak{r}^b$ satisfies equation (8.58).

Note that equation (8.58) coincide with equation (A.2.32) if we write $\boldsymbol{\Phi}$ for $\mathfrak{r}^b$ and $\boldsymbol{E}$ for $\boldsymbol{b}$. Consequently, the response function $\mathfrak{r}^b$ describes isotropic material behavior.

According to the representation theorem of isotropic tensor valued tensor functions (A.2.50) it follows from equation (8.58) that the Cauchy stress tensor is quadratic polynomial of the Cauchy strain tensor $\boldsymbol{b}$:

$$\boldsymbol{t} = \alpha_0 \boldsymbol{1} + \alpha_1 \boldsymbol{b} + \alpha_2 \boldsymbol{b}^2, \tag{8.59}$$

where the scalars α_0, α_1 and α_2 are functions of the scalar invariants b_I, b_{II} and b_{III}: $\alpha_0 = \alpha_0(b_I, b_{II}, b_{III})$, $\alpha_1 = \alpha_1(b_I, b_{II}, b_{III})$, $\alpha_2 = \alpha_2(b_I, b_{II}, b_{III})$.

Remark 8.13: Note that the representation of the Cauchy stress tensor by equation (8.59) is clearly independent of the observers: it satisfies the requirement of objectivity.

Dot multiply equation (8.59) by

$$\det(\boldsymbol{U})\, \boldsymbol{F}^{-1} = \det(\sqrt{\boldsymbol{C}})\, \boldsymbol{F}^{-1} = \sqrt{C_{III}}\, \boldsymbol{F}^{-1} = J \boldsymbol{F}^{-1}$$

from left and $\boldsymbol{F}^{-T}$ from right then take into account that $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ while $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$. We get

$$J \boldsymbol{F}^{-1} \cdot \boldsymbol{t} \cdot \boldsymbol{F}^{-T} \underset{(5.27)}{=\uparrow=} \boldsymbol{S}(\boldsymbol{C}) = \breve{\alpha}_0 \boldsymbol{C}^{-1} + \breve{\alpha}_1 \boldsymbol{1} + \breve{\alpha}_2 \boldsymbol{C}\,, \tag{8.60}$$

where $\breve{\alpha}_\ell = \sqrt{C_{III}}\, \alpha_\ell(C_I, C_{II}, C_{III})$ $(\ell = 0, 1, 2)$ since the scalar invariants of $\boldsymbol{C}$ and $\boldsymbol{b}$ are the same. Equation (8.60) shows that the second Piola-Kirchhoff stress tensor for an isotropic body is the function of the right Cauchy-Green tensor $\boldsymbol{C}$.

By recalling the representation theorem of isotropic tensor valued tensor functions (A.2.50) we can also come to the conclusion that

$$\boldsymbol{S}(\boldsymbol{C}) = \beta_0 \boldsymbol{1} + \beta_1 \boldsymbol{C} + \beta_2 \boldsymbol{C}^2\,, \tag{8.61}$$

in which the coefficients β_0, β_1 and β_2 are functions of the scalar invariants C_I, C_{II} and C_{III}: $\beta_0 = \beta_0(C_I, C_{II}, C_3)$, $\beta_1 = \beta_1(C_I, C_{II}, C_3)$, $\beta_2 = \beta_2(C_I, C_{II}, C_3)$.

8.5. Hyperelastic materials

8.5.1. Thermomechanical behavior. A solid body is said to be hyperelastic (or Green elastic) if there exist an internal energy function per unit mass $e = e^\circ$. The Helmholtz free energy [27] per unit mass is defined by the following equation

$$f = e - \Theta s\,. \tag{8.62}$$

Remark 8.14: For elastic bodies the internal energy depends, among others, on the components of the strain tensors, the entropy depends on the temperature and also on the components of the strain tensors. Hence the Helmholtz free energy is independent of the temperature gradient $\Theta\nabla$.

Making use of f the dissipation power for a unit volume given by equation (6.77) can be rewritten into the form

$$\phi_D = \boldsymbol{t} \cdot\cdot\, \boldsymbol{d} - \rho((e)^{\bullet} - \Theta\,(s)^{\bullet}) = \boldsymbol{t} \cdot\cdot\, \boldsymbol{d} - \rho((f)^{\bullet} + s\,(\Theta)^{\bullet}) \geq 0\,. \tag{8.63}$$

With (8.63) for the local form of the entropy theorem (6.78) we get:

$$\boldsymbol{t} \cdot\cdot\, \boldsymbol{d} - \rho((f)^{\bullet} + s\,(\Theta)^{\bullet}) - \frac{\mathbf{q} \cdot (\Theta \nabla)}{\Theta} \geq 0\,. \tag{8.64}$$

Needles to emphasize that the thermomechanical material models take the heat effects into account.

Note that the mechanical state of the body is determined by two different variable groups.

(i) The first group is constituted by those variables being formally independent of the material parameters included in the constitutive equations and reflecting, therefore, the material behavior of the body. These variables include the motion law $\mathbf{x} = \boldsymbol{\chi}\,(\mathbf{X}; t)$ – see (2.2) – and the quantities that can be obtained from the motion law as, for example, the strain tensors and the strain rate tensor. They describe the deformation state of the body and its change with time. Since the heat effects are also to be taken into account the temperature distribution $\Theta\,(\mathbf{X}; t)$ also belongs to this group of variables.

(ii) The second group is constituted by those variables which formally depend on the material parameters of the constitutive relations. These include, for instance, the stress tensors $\boldsymbol{t}$, $\boldsymbol{P}$, $\boldsymbol{S}$, the internal energy e, the entropy s (consequently, the Helmholtz free energy f as well) and the heat flux vector $\mathbf{q}$.

Since the body considered is elastic the dissipation power is zero:

$$\phi_D = \boldsymbol{t} \cdot\cdot\, \boldsymbol{d} - \rho((f)^{\bullet} + s\,(\Theta)^{\bullet}) = 0\,. \tag{8.65}$$

A comparison of equations (6.59) and (6.8) leads to the following result:

$$J\,\boldsymbol{d} \cdot\cdot\, \boldsymbol{t} = \frac{\rho^{\circ}}{\rho}\,\boldsymbol{d} \cdot\cdot\, \boldsymbol{t} = \boldsymbol{S} \cdot\cdot\, (\boldsymbol{E})^{\bullet}\,. \tag{8.66}$$

Hence

$$\frac{\rho^{\circ}}{\rho}\,\phi_D = \boldsymbol{S} \cdot\cdot\, (\boldsymbol{E})^{\bullet} - \rho^{\circ}((f)^{\bullet} + s\,(\Theta)^{\bullet}) = 0\,. \tag{8.67}$$

With regard to Remark 8.14 the material time derivative of the Helmholtz free energy assumes the form:

$$(f)^{\bullet} = \frac{\partial f}{\partial \boldsymbol{E}} \cdot\cdot\, (\boldsymbol{E})^{\bullet} + \frac{\partial f}{\partial \Theta}\,(\Theta)^{\bullet} \quad \text{or} \quad (f)^{\bullet} = \frac{\partial f}{\partial E_{AB}}\,(E_{AB})^{\bullet} + \frac{\partial f}{\partial \Theta}\,(\Theta)^{\bullet}\,. \tag{8.68}$$

Calculating now the material time derivative of equation (8.69) we may write

$$(e)^{\bullet} = (f)^{\bullet} + (\Theta)^{\bullet}\,s + (s)^{\bullet}\,\Theta\,. \tag{8.69}$$

Substituting (8.68) and (8.69) into (8.67) yields

$$\left(\rho^{\circ}\frac{\partial f^{\circ}}{\partial \boldsymbol{E}} - \boldsymbol{S}\right) \cdot\cdot\, (\boldsymbol{E})^{\bullet} + \rho^{\circ}\left(\frac{\partial f^{\circ}}{\partial \Theta} + s\right)(\Theta)^{\bullet} = 0 \tag{8.70a}$$

or

$$\left(\rho^\circ \frac{\partial f^\circ}{\partial E_{AB}} - S_{AB}\right)(E_{AB})^{\cdot} + \rho^\circ \left(\frac{\partial f^\circ}{\partial \Theta} + s^\circ\right)(\Theta)^{\cdot} = 0. \tag{8.70b}$$

where $f = f^\circ$ – the Helmholtz free energy is for a unit mass, therefore it is the same both in material and spatial descriptions. The same is valid for the entropy and the internal energy as well: $s = s^\circ$ and $e = e^\circ$ – see equation (6.54) with the relation for e° which is valid formally for f and s too.

Since equations (8.70) should be fulfilled for any $(\boldsymbol{E})^{\cdot}$ and $(\Theta)^{\cdot}$ it follows with regard to (8.62) that

$$\boxed{\boldsymbol{S} = \rho^\circ \frac{\partial f^\circ}{\partial \boldsymbol{E}} = \rho^\circ \left(\frac{\partial e^\circ}{\partial \boldsymbol{E}} - \Theta \frac{\partial s^\circ}{\partial \boldsymbol{E}}\right) \quad \text{or} \quad S_{AB} = \rho^\circ \frac{\partial f^\circ}{\partial E_{AB}} = \rho^\circ \left(\frac{\partial e^\circ}{\partial E_{AB}} - \Theta \frac{\partial s^\circ}{\partial E_{AB}}\right)} \tag{8.71}$$

and

$$\boxed{s^\circ = -\frac{\partial f^\circ}{\partial \Theta}.} \tag{8.72}$$

Equations (8.71) and (8.72) are material equations (or constitutive equations) for the second Piola-Kirchhoff stress tensor and the entropy in material description.

REMARK 8.15: If the temperature is constant – there is no heat effect – the entropy s° is zero and equation (8.71) coincides with equation (6.67).

If we write f for e in equation (6.68) we get

$$\boldsymbol{F} \cdot \frac{\partial f^\circ}{\partial \boldsymbol{E}} \cdot \boldsymbol{F}^T = \frac{\partial e}{\partial \boldsymbol{e}} \quad \text{or} \quad F_{rA} \frac{\partial e^\circ}{\partial E_{AB}} F_{Bs} = \frac{\partial f}{\partial e_{k\ell}}. \tag{8.73}$$

Dot multiplying equation (8.71) by $\rho \boldsymbol{F}/\rho^\circ$ from left and $\boldsymbol{F}^T$ from right then taking equations (6.69) and (8.73) into account yield

$$\boxed{\boldsymbol{t} = \rho \frac{\partial f}{\partial \boldsymbol{e}} = \rho \left(\frac{\partial e}{\partial \boldsymbol{e}} - \Theta \frac{\partial s}{\partial \boldsymbol{e}}\right) \quad \text{or} \quad t_{k\ell} = \rho \frac{\partial f}{\partial e_{k\ell}} = \rho \left(\frac{\partial e}{\partial e_{k\ell}} - \Theta \frac{\partial s}{\partial e_{k\ell}}\right).} \tag{8.74}$$

This equation is similar to equation (8.71). It is de facto the material equation (or constitutive equation) for the Cauchy stress tensor.

It is assumed for equation (8.71) that the Helmholtz free energy, the internal energy and the entropy are known in the forms $f^\circ = f^\circ(\boldsymbol{E}, \Theta)$, $e^\circ = e^\circ(\boldsymbol{E}, \Theta)$ and $s^\circ = s^\circ(\boldsymbol{E}, \Theta)$. As regards equation (8.74) it is also assumed that $f = f(\boldsymbol{e}, \Theta)$, $e = e(\boldsymbol{e}, \Theta)$ and $s = s(\boldsymbol{e}, \Theta)$.

REMARK 8.16: If the temperature is constant – there is no heat effect – the entropy s is zero and equation (8.74) coincides with equation (6.70).

REMARK 8.17: Equations (8.71) and (8.72) are equivalent to 7 scalar equations. If the heat effects are taken into account the number of missing equations is 11. This means that 4 equations are still missing. The first missing equation concerns the Helmholtz free energy $f^\circ = f^\circ(\boldsymbol{E}, \Theta)$, while the other three missing equations should give the heat flux vector as a function of the material parameters included in the corresponding constitutive equations.

8.5.2. Heat conduction equation. Since the heat processes have an influence on the mechanical processes and conversely the mechanical processes affect the heat processes it is worth clarifying what equation governs the heat conduction in solids. This issue is not closely related to the issue of material equations (constitutive equations), but its importance makes it worth reviewing.

It follows from equation (8.67) that

$$\boldsymbol{S}\cdot\cdot(\boldsymbol{E})^{\cdot} = -\rho^\circ((e^\circ)^{\cdot} - \Theta\,(s^\circ)^{\cdot})\,.$$

Substituting it into the energy equation (6.62) yields

$$\mathbf{q}^\circ\cdot\nabla^\circ + \rho^\circ\Theta\,(s^\circ)^{\cdot} - \rho^\circ h^\circ = 0\,, \tag{8.75}$$

where according to (8.72) it holds that

$$(s^\circ)^{\cdot} = -\left(\frac{\partial f^\circ}{\partial\Theta}\right)^{\cdot} = -\frac{\partial^2 f^\circ}{\partial\Theta\partial\boldsymbol{E}}\cdot\cdot(\boldsymbol{E})^{\cdot} - \frac{\partial^2 f^\circ}{\partial\Theta^2}\,(\Theta)^{\cdot}\,.$$

Inserting this equation into (8.75) we get

$$\mathbf{q}^\circ\cdot\nabla^\circ - \rho^\circ\Theta\left[\frac{\partial^2 f^\circ}{\partial\Theta\partial\boldsymbol{E}}\cdot\cdot(\boldsymbol{E})^{\cdot} + \frac{\partial^2 f^\circ}{\partial\Theta^2}\,(\Theta)^{\cdot}\right] - \rho^\circ h^\circ = 0 \tag{8.76}$$

which is the heat conduction equation in material description. For the material equations and the heat conduction equation, we shall restrict our attention to the linear theory in the following.

REMARK 8.18: Equation (8.76) should be supplemented by the relationship between the heat flux vector and temperature and the free energy equation $f^\circ = f^\circ(\boldsymbol{E},\Theta)$ – these are missing material equations (constitutive equations). This will also be performed within the framework of the linear theory.

8.5.3. Linearization. In the linear theory the tensor $\boldsymbol{\varepsilon}$ – see (2.39), (2.55) or (4.9) – corresponds to the tensor $\boldsymbol{E}$, while, according to (5.31), the stresses are given by the tensor $\boldsymbol{\sigma}$. Let

$$\boldsymbol{\kappa} = \kappa_{k\ell}\mathbf{i}_k\circ\mathbf{i}_\ell \tag{8.77}$$

be the heat conduction tensor (or the tensor of heat conductivity) [70]. The material equations needed are as follows:

$$f = f(\Theta,\boldsymbol{\varepsilon}), \qquad s = -\frac{\partial f}{\partial\Theta}, \tag{8.78}$$

$$\boldsymbol{\sigma} = \rho^\circ\frac{\partial f}{\partial\boldsymbol{\varepsilon}}, \qquad \mathbf{q} = -\boldsymbol{\kappa}\cdot(\Theta\nabla^\circ)\,. \tag{8.79}$$

Note that equation $(8.79)_2$ is the Fourier law of heat conduction [23].

We remark that superscript $^\circ$ is omitted here and later on as well since we make no difference between the initial and current states of the body within the framework of the linear theory. The only exception to this rule is the density.

After substituting (8.79) into $(8.76)_2$ we have

$$\nabla\cdot[\boldsymbol{\kappa}\cdot(\Theta\nabla)] + \rho^\circ\Theta\left[\frac{\partial^2 f}{\partial\Theta\partial\boldsymbol{\varepsilon}}\cdot\cdot(\boldsymbol{\varepsilon})^{\cdot} + \frac{\partial^2 f}{\partial\Theta^2}\,(\Theta)^{\cdot}\right] + \rho^\circ h = 0\,. \tag{8.80}$$

Since the dissipation power Φ_D is zero it follows from equation (6.78) that

$$-\mathbf{q}\cdot(\Theta\nabla) = \underset{(8.79)_2}{\uparrow} = -(\Theta\nabla)\cdot\boldsymbol{\kappa}\cdot(\Theta\nabla) = \boldsymbol{\kappa}\cdot\cdot\,[(\Theta\nabla)\circ(\Theta\nabla)] \geq 0\,. \tag{8.81}$$

Note that the tensor $(\Theta\nabla^{\circ})\circ(\Theta\nabla^{\circ})$ is symmetric; it is, therefore, obvious that the skew part of the tensor $\boldsymbol{\kappa}$ has no effect on the value of the above energy product. For this reason it is customary to assume that the tensor $\boldsymbol{\kappa}$ is symmetric [67]. If the body is isotropic the tensor $\boldsymbol{\kappa}$ is also isotropic, i.e., it has the form

$$\boldsymbol{\kappa} = \kappa\boldsymbol{1}\,, \tag{8.82}$$

where the scalar κ is a function of Θ. Then equation $(8.79)_2$ takes the form

$$\mathbf{q} = \kappa(\Theta\nabla), \tag{8.83}$$

The scalar κ is called heat conduction coefficient.

It is worth emphasizing the following:

- The internal energy e is independent of the entropy s.
- The Helmholtz free energy f is independent of the temperature gradient $\Theta\nabla$.
- Due to the fulfillment of the material equations $(8.78)_{1,2}$ and $(8.79)_1$ it is zero the dissipation power Φ_D. This is a fundamental feature of the thermoelastic behavior.

Further simplifications can be achieved within the framework of the linear thermoelasticity if not only the strains are assumed to be small but the temperature changes as well. Expend the Helmholtz free energy $f(\Theta,\boldsymbol{\varepsilon})$ into Taylor series in the neighborhood of the points $\Theta = \Theta_{\circ}$ and $\boldsymbol{\varepsilon} = \mathbf{0}$. We get

$$f(\Theta,\boldsymbol{\varepsilon}) = f(\Theta_{\circ},\mathbf{0}) + \frac{\partial f(\Theta_{\circ},\mathbf{0})}{\partial\Theta}(\Theta-\Theta_{\circ}) + \frac{\partial f(\Theta_{\circ},\mathbf{0})}{\partial\boldsymbol{\varepsilon}}\cdot\boldsymbol{\varepsilon} +$$
$$+\frac{1}{2}\left[\boldsymbol{\varepsilon}\cdot\cdot\frac{\partial^2 f(\Theta_{\circ},\mathbf{0})}{\partial\boldsymbol{\varepsilon}^2}\cdot\cdot\boldsymbol{\varepsilon} + 2\frac{\partial^2 f(\Theta_{\circ},\mathbf{0})}{\partial\boldsymbol{\varepsilon}\partial\Theta}\cdot\cdot\boldsymbol{\varepsilon}\,(\Theta-\Theta_{\circ}) + \frac{\partial^2 f(\Theta_{\circ},\mathbf{0})}{\partial\Theta^2}(\Theta-\Theta_{\circ})^2\right] +$$
$$+\ldots \tag{8.84}$$

We may assume that the initial value of the Helmholtz free energy can be dropped since equations $(8.78)_2$ and $(8.79)_1$ contain its derivatives only. As regards the first derivatives it holds on the basis of the previous two equations that

$$s(\Theta_{\circ},\mathbf{0}) = -\frac{\partial f(\Theta_{\circ},\mathbf{0})}{\partial\Theta} \qquad \text{and} \qquad \boldsymbol{\sigma}(\Theta_{\circ},\mathbf{0}) = \rho^{\circ}\frac{\partial f(\Theta_{\circ},\mathbf{0})}{\partial\boldsymbol{\varepsilon}}\,. \tag{8.85}$$

The initial value $s(\Theta_{\circ},\mathbf{0})$ of the Helmholtz strain energy and the initial value $\boldsymbol{\sigma}(\Theta_{\circ},\mathbf{0})$ of the stress field can be set to zero for a number of problems in thermoelasticity. In what follows we shall assume that they can be neglected.

Within the framework of the geometrically and physically linear theory of thermoelasticity the derivatives of order greater than two are all left out of consideration in the Taylor series (8.84) since they describe non-linear material behavior.

Let us introduce the following notations:

(i)

$$^{(4)}\boldsymbol{C} = \rho^\circ \boldsymbol{\varepsilon} \cdot\cdot \frac{\partial^2 f(\Theta_\circ, \mathbf{0})}{\partial \boldsymbol{\varepsilon}^2}; \quad \mathcal{C}_{mnk\ell} = \rho^\circ \frac{\partial^2 f(\Theta_\circ, \mathbf{0})}{\partial \varepsilon_{mn} \partial \varepsilon_{k\ell}} . \tag{8.86}$$

This fourth order tensor is that of the elastic coefficients. Note that $\mathcal{C}_{mnk\ell} = \mathcal{C}_{mn\ell k} = \mathcal{C}_{nm\ell k} = \mathcal{C}_{nmk\ell}$ which means that $^{(4)}\boldsymbol{C}$ is symmetric in respect of the index pairs mn and $k\ell$.

(ii)

$$\boldsymbol{\beta} = -\rho^\circ \boldsymbol{\varepsilon} \cdot\cdot \frac{\partial^2 f(\Theta_\circ, \mathbf{0})}{\partial \boldsymbol{\varepsilon} \partial \Theta}; \quad \beta_{k\ell} = \beta_{\ell k} = -\rho^\circ \frac{\partial^2 f(\Theta_\circ, \mathbf{0})}{\partial \varepsilon_{k\ell} \partial \Theta} . \tag{8.87}$$

This symmetric tensor gives the stresses caused by a unit temperature change.

(iii)

$$c = -\rho^\circ \boldsymbol{\varepsilon} \cdot\cdot \frac{\partial^2 f(\Theta_\circ, \mathbf{0})}{\partial \Theta^2} . \tag{8.88}$$

This quantity is called heat capacity for a unit volume, i.e., it is the amount of heat that must be added to a unit volume of the substance in order to cause an increase of one unit in temperature.

Substituting equations (8.86), (8.87) and (8.88) into the Taylor series (8.84) and taking into account what has been said about the initial values of f, s and $\boldsymbol{\sigma}$ we get

$$\rho^\circ f(\Theta, \boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon} \cdot\cdot {}^{(4)}\boldsymbol{C} \cdot\cdot \boldsymbol{\varepsilon} - \boldsymbol{\beta} \cdot\cdot \boldsymbol{\varepsilon} \, (\Theta - \Theta_\circ) - \frac{1}{2} c \, (\Theta - \Theta_\circ)^2 \tag{8.89a}$$

or

$$\rho^\circ f(\Theta, \varepsilon_{pq}) = \frac{1}{2} \varepsilon_{mn} \mathcal{C}_{mnk\ell} \varepsilon_{k\ell} - \beta_{k\ell} \varepsilon_{k\ell} \, (\Theta - \Theta_\circ) - \frac{1}{2} c \, (\Theta - \Theta_\circ)^2 . \tag{8.89b}$$

Making use of (8.89) the following material equations (constitutive equations) can be obtained from $(8.78)_2$ and $(8.79)_1$:

$$\boxed{\rho^\circ s = -\rho^\circ \frac{\partial f(\Theta, \boldsymbol{\varepsilon})}{\partial \Theta} = \boldsymbol{\beta} \cdot\cdot \boldsymbol{\varepsilon} + c \, (\Theta - \Theta_\circ) \, ,} \tag{8.90}$$

$$\boxed{\boldsymbol{\sigma} = \rho^\circ \frac{\partial f(\Theta, \boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = {}^{(4)}\boldsymbol{C} \cdot\cdot \boldsymbol{\varepsilon} - \boldsymbol{\beta} \, (\Theta - \Theta_\circ) \, .} \tag{8.91}$$

Let the tensor $^{(4)}\boldsymbol{S}$ be the inverse of the tensor $^{(4)}\boldsymbol{C}$ – we remind the reader of Subsection 1.4.6.3 here. Then it holds that

$$^{(4)}\boldsymbol{S} \cdot\cdot {}^{(4)}\boldsymbol{C} = {}^{(4)}\boldsymbol{1} , \qquad \mathcal{S}_{pqmn} \, \mathcal{C}_{mnk\ell} = \delta_{pqk\ell} = \underset{(1.162)}{\uparrow} = \delta_{pk} \delta_{q\ell} \tag{8.92}$$

The fourth order unit tensor $^{(4)}\boldsymbol{1}$ maps each second order tensor into itself:

$$^{(4)}\boldsymbol{S} \cdot\cdot {}^{(4)}\boldsymbol{C} \cdot\cdot \boldsymbol{\varepsilon} = {}^{(4)}\boldsymbol{1} \cdot\cdot \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} , \quad \mathcal{S}_{pqmn} \, \mathcal{C}_{mnk\ell} \, \varepsilon_{k\ell} = \delta_{pk} \delta_{q\ell} \varepsilon_{k\ell} = \varepsilon_{pq} \tag{8.93}$$

Let us multiply equation (8.91) by ${}^{(4)}\boldsymbol{S}$. Taking (8.93) into account we get the inverse of the material equation (constitutive equation) (8.91):

$$\boldsymbol{\varepsilon} = {}^{(4)}\boldsymbol{S}\cdot\cdot\,\boldsymbol{\sigma} + \boldsymbol{\alpha}\,(\Theta - \Theta_\circ)\ , \tag{8.94}$$

where

$$\boldsymbol{\alpha} = {}^{(4)}\boldsymbol{S}\cdot\cdot\,\boldsymbol{\beta}\,, \qquad \alpha_{pq} = \mathcal{S}_{pqmn}\beta_{mn}\,. \tag{8.95}$$

Since $\mathcal{S}_{pqmn} = \mathcal{S}_{qpmn}$ – see Problem 8.4 – the tensor α_{pq} is also symmetric. Its name is tensor of the thermal expansion coefficients since it gives the strains due to a unit change in the temperature.

Using equations (8.86), (8.87) and (8.88) we can transform the heat conduction equation (8.80) into the following form:

$$\nabla\cdot[\boldsymbol{\kappa}\cdot(\Theta\nabla)] - \Theta\boldsymbol{\beta}\cdot\cdot\,\boldsymbol{\epsilon} - c\Theta\,(\Theta)^{\boldsymbol{\cdot}} = -\rho^\circ h\,. \tag{8.96}$$

For small temperature changes it holds that

$$\Theta = \Theta_\circ + \vartheta = \Theta_\circ\left(1 + \frac{\vartheta}{\Theta_\circ}\right), \qquad \frac{\vartheta}{\Theta_\circ} \ll 1\,, \tag{8.97}$$

where $\Theta_\circ$ is usually 273 K°. Utilizing this relationship equation (8.96) can be linearized:

$$\nabla\cdot[\boldsymbol{\kappa}\cdot(\vartheta\nabla)] - \Theta\boldsymbol{\beta}\cdot\cdot\,\boldsymbol{\epsilon} - c\Theta_\circ\,(\vartheta)^{\boldsymbol{\cdot}} = -\rho^\circ h\,. \tag{8.98}$$

Let us assume that the material is isotropic. For isotropic material it follows from relations (1.164) and (1.169a), (1.169b) that

$$\boldsymbol{\beta} = \beta\boldsymbol{1} \tag{8.99}$$

and

$${}^{(4)}\boldsymbol{C} = 2\mu{}^{(4)}\boldsymbol{1} + \lambda\boldsymbol{1}\circ\boldsymbol{1} = 2\mu\left({}^{(4)}\boldsymbol{1} + \frac{\nu}{1-2\nu}\boldsymbol{1}\circ\boldsymbol{1}\right), \tag{8.100a}$$

$${}^{(4)}\boldsymbol{S} = \frac{1}{2\mu}\left({}^{(4)}\boldsymbol{1} - \frac{\lambda}{3\lambda+2\mu}\boldsymbol{1}\circ\boldsymbol{1}\right) = \frac{1}{2\mu}\left({}^{(4)}\boldsymbol{1} - \frac{\nu}{1+\nu}\boldsymbol{1}\circ\boldsymbol{1}\right), \tag{8.100b}$$

where β is the stress due to a unit change in the temperature, $\mu(\Theta)$ and $\lambda(\Theta)$ and are called Lamé[2] numbers (or constants) [10, 14] while ν is the Poisson[3] ratio. In engineering practice μ is also referred to as shear modulus of elasticity and is denoted by $G = \mu$.

Making use of (8.99) for $\boldsymbol{\beta}$ and (8.100a) for ${}^{(4)}\boldsymbol{C}$ we can transform formula (8.89a) of the Helmholtz free energy into the following form:

$$\rho^\circ f(\Theta,\boldsymbol{\varepsilon}) = \mu\boldsymbol{\varepsilon}\cdot\cdot\,\boldsymbol{\varepsilon} + \frac{1}{2}\lambda(\varepsilon_I)^2 - \beta\varepsilon_I\,(\Theta-\Theta_\circ) - \frac{1}{2}c\,(\Theta-\Theta_\circ)^2\,, \tag{8.101}$$

where $\varepsilon_I = \boldsymbol{\varepsilon}\cdot\cdot\boldsymbol{1}$ is the first scalar invariant of the tensor $\boldsymbol{\varepsilon}$ and β is the stress change due to a unit change in the temperature.

Substituting (8.99) and (8.100a) into (8.90) yields the following material equations

$$\rho^\circ s = -\rho^\circ\frac{\partial f(\Theta,\boldsymbol{\varepsilon})}{\partial\Theta} = \beta\varepsilon_I + c\,(\Theta-\Theta_\circ)\ , \tag{8.102}$$

[2] Baptiste Lamé, 1795–1870
[3] Siméon Denis Poisson, 1781-1840

and

$$\boxed{\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\varepsilon_I\boldsymbol{1} - \beta\,(\Theta - \Theta_\circ)\,\boldsymbol{1}} \tag{8.103a}$$

or

$$\boxed{\boldsymbol{\sigma} = 2\mu\left(\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\varepsilon_I\boldsymbol{1}\right) - \beta\,(\Theta - \Theta_\circ)\,\boldsymbol{1}\,.} \tag{8.103b}$$

With (8.100b) for ${}^{(4)}\boldsymbol{S}$ we obtain from (8.94), (8.95) and (8.99) the inverses of the material equations (8.103):

$$\boxed{\boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{3\lambda+2\mu}\sigma_I\boldsymbol{1}\right) + \frac{\beta}{3\lambda+2\mu}\,(\Theta - \Theta_\circ)\,\boldsymbol{1}} \tag{8.104a}$$

or

$$\boxed{\boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\nu}{1+\nu}\sigma_I\boldsymbol{1}\right) + \alpha\,(\Theta - \Theta_\circ)\,\boldsymbol{1}\,,} \tag{8.104b}$$

where

$$\alpha = \frac{\beta}{3\lambda+2\mu} = \frac{1}{2\mu}\frac{1-2\nu}{1+\nu}\beta \tag{8.105}$$

is the thermal expansion coefficient. The scalar invariants are given by:

$$\sigma_I = (3\lambda+2\mu)\varepsilon_I - 3\beta\,(\Theta - \Theta_\circ) = 2\mu\frac{1+\nu}{1-2\nu}\varepsilon_I - 3\beta\,(\Theta - \Theta_\circ)\,, \tag{8.106a}$$

$$\varepsilon_I = \frac{\sigma_I}{3\lambda+2\mu} + \frac{3\beta}{3\lambda+2\mu}\,(\Theta - \Theta_\circ) = \frac{1}{2\mu}\frac{1-2\nu}{1+\nu}\sigma_1 + 3\alpha\,(\Theta - \Theta_\circ)\,. \tag{8.106b}$$

The modulus of elasticity E is also a material parameter:

$$E = 2\mu\,(1+\nu)\,, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}\,. \tag{8.107}$$

Remark 8.19: The number of independent components in $\mathcal{C}_{mnk\ell}$ is not 81 but 21 only. The proof of this statement is left for Problem 8.3.

Remark 8.20: As regards the isotropic bodies the number of independent elasticity parameters is only two, eg. λ and μ, or $G = \mu$ and ν, or E and ν.

If the body is isotropic $\boldsymbol{\kappa} = \kappa\boldsymbol{1}$, $\boldsymbol{\beta} = \beta\boldsymbol{1}$ – see equations (8.82) and (8.99) – and the temperature change is small the heat conduction equation (8.98) assumes the form

$$\nabla\cdot[\kappa\cdot(\vartheta\nabla)] - \Theta\beta\,(\epsilon_I)^{\bullet} - c\Theta_\circ\,(\vartheta)^{\bullet} = -\rho^\circ h\,. \tag{8.108}$$

8.5.4. Elastic bodies with finite deformations. If the body under consideration is Green elastic, the heat effects can be neglected and the deformations are finite the possible forms of the constitutive relations are given by (6.67) and (6.99) for the second and first Piola-Kirchhoff stress tensors:

$$\boldsymbol{S} = \rho^\circ \frac{\partial e^\circ}{\partial \boldsymbol{E}}\,, \quad S_{AB} = \rho^\circ \frac{\partial e^\circ}{\partial E_{AB}}; \qquad \boldsymbol{P} = \rho^\circ \frac{\partial e^\circ}{\partial \boldsymbol{F}}\,, \quad P_{kL} = \rho^\circ \frac{\partial e^\circ}{\partial F_{kL}}\,. \tag{8.109}$$

Note that e° might be regarded as if it were the function of $\boldsymbol{E}$ and $\boldsymbol{F}$ as well.

The product $\psi = \rho^\circ e^\circ$ is the strain energy density for a unit volume.

We shall assume, in accordance with the paragraph that follows equation (8.85), that $\psi(\boldsymbol{F})$ vanishes in the initial configuration (in the non-deformed body). Since then $\boldsymbol{F} = \boldsymbol{1}$ we have:

$$\psi|_{\boldsymbol{F}=\boldsymbol{1}} = 0\,. \tag{8.110}$$

It is also a natural requirement that the strain energy density for a unit volume shoul be equal to either zero or a positive quantity if the body considered is deformed:

$$\psi(\boldsymbol{F}) \geq 0\,. \tag{8.111}$$

This means that ψ has a global minimum if $\boldsymbol{F} = \boldsymbol{1}$, i.e., it holds that

$$\psi(\boldsymbol{F}) > \psi(\boldsymbol{1}) = 0\,, \quad \forall\, \boldsymbol{F} \neq \boldsymbol{1} \tag{8.112}$$

Suppose that the compressive load on the body is such that the body shrinks to a single point. Then the Jacobian must be equal to zero. However, strain energy of infinite magnitude is required for this process to occur. Consequently,

$$\psi(\boldsymbol{F}) \to +\infty \quad \text{if} \quad J = \det(\boldsymbol{F}) \to 0^+. \tag{8.113a}$$

If, however, the load on the body is such that its volume tends to infinity, then the Jacobian also tends to infinity. This process also requires a strain energy of infinite magnitude. That is

$$\psi(\boldsymbol{F}) \to +\infty \quad \text{if} \quad J = \det(\boldsymbol{F}) \to +\infty. \tag{8.113b}$$

Note that ψ should be independent of the observer. This means that the following equation is to be satisfied for any $\boldsymbol{\mathcal{T}}$:

$$\psi(\boldsymbol{F}') = \psi(\boldsymbol{F}) = \psi(\boldsymbol{\mathcal{T}} \cdot \boldsymbol{F}')\,, \tag{8.114}$$

where $\boldsymbol{F}' = \boldsymbol{R}' \cdot \boldsymbol{U}'$ from the polar decomposition theorem. This equation should be valid for any orthogonal tensor, therefore, if we write $\boldsymbol{R}'^T$ for $\boldsymbol{\mathcal{T}}$ we get,

$$\psi(\boldsymbol{F}') = \psi(\underbrace{\boldsymbol{R}'^T \cdot \boldsymbol{R}'}_{\boldsymbol{1}} \cdot \boldsymbol{U}') = \psi(\boldsymbol{U}')\,. \tag{8.115}$$

This means that ψ is a function of the right stretch tensor $\boldsymbol{U}$. With regard to relations (2.32) we may also assume that $\psi = \psi(\boldsymbol{E})$ or $\psi = \psi(\boldsymbol{C})$.

In the sequel we restrict our attention to isotropic material behavior only.

8.5.5. Compressible materials. If the body under consideration is isotropic it follows from all that has been said in Subsection A.2.2 that the strain energy function depends on the scalar invariants of the Green-Lagrange strain tensor $\boldsymbol{E}$ or those of the right Cauchy-Green tensor $\boldsymbol{C}$.

Assume first that $\psi = \psi(E_I, E_{II}, E_{III}) = \rho^\circ e(E_I, E_{II}, E_{III})$ in equations (6.67), $(8.109)_{1,2}$. Then

$$S_{AB} = \frac{\partial \psi}{\partial E_{AB}} = \frac{\partial \psi}{\partial E_I}\frac{\partial E_I}{\partial E_{AB}} + \frac{\partial \psi}{\partial E_{II}}\frac{\partial E_{II}}{\partial E_{AB}} + \frac{\partial \psi}{\partial E_{III}}\frac{\partial E_{III}}{\partial E_{AB}} . \tag{8.116}$$

where on the basis of (1.113a), (1.113b) and (1.130)

$$\begin{aligned} E_I = E_{KK}\,, \qquad E_{II} = \frac{1}{2}\left(E_I^2 - E_{KL}E_{LK}\right)\,, \\ E_{III} = \frac{1}{6}\left(-2E_I^3 + 6E_IE_{II} + 2E_{KP}E_{PQ}E_{QK}\right)\,. \end{aligned} \tag{8.117}$$

Consequently,

$$\frac{\partial E_I}{\partial E_{AB}} = \delta_{AB}\,, \tag{8.118a}$$

$$\frac{\partial E_{II}}{\partial E_{AB}} = E_I\frac{\partial E_I}{\partial E_{AB}} - \frac{\partial E_{KL}}{\partial E_{AB}}E_{LK} = E_I\delta_{AB} - \delta_{KA}\delta_{LB}E_{KL} = E_I\delta_{AB} - E_{AB} \tag{8.118b}$$

and

$$\begin{aligned} \frac{\partial E_{III}}{\partial E_{AB}} &= -E_I^2\delta_{AB} + E_{II}\delta_{AB} + E_I\left(E_I\delta_{AB} - E_{AB}\right) + \\ &+ \frac{1}{3}\left(\delta_{KA}\delta_{PB}E_{PQ}E_{QK} + E_{KP}\delta_{QA}\delta_{QB}E_{QK} + E_{KP}E_{PQ}\delta_{QA}\delta_{KB}\right) = \\ &= E_{II}\delta_{AB} - E_IE_{AB} + \frac{1}{3}\left(E_{AQ}E_{QB} + E_{AK}E_{KB} + E_{AP}E_{PB}\right) = \\ &\underset{K=Q,}{=} \uparrow \underset{P=Q,}{=} E_{II}\delta_{AB} - E_IE_{AB} + E_{AQ}E_{QB}\,. \end{aligned} \tag{8.118c}$$

After substituting these derivatives into equation (8.116) we get the coefficients in the constitutive equation of a non-linearly elastic and isotropic body:

$$\boxed{S_{AB} = a_0^\circ\delta_{AB} + a_1^\circ E_{AB} + a_2^\circ E_{AQ}E_{QB}\,; \qquad \boldsymbol{S} = a_0^\circ\boldsymbol{1} + a_1^\circ\boldsymbol{E} + a_2^\circ\boldsymbol{E}^2\,,} \tag{8.119}$$

where

$$a_0^\circ = \frac{\partial \psi}{\partial E_I} + E_I\frac{\partial \psi}{\partial E_{II}} + E_{II}\frac{\partial \psi}{\partial E_{III}}\,, \tag{8.120a}$$

$$a_1^\circ = -\frac{\partial \psi}{\partial E_{II}} - E_I\frac{\partial \psi}{\partial E_{III}}\,, \qquad a_2^\circ = \frac{\partial \psi}{\partial E_{III}}\,. \tag{8.120b}$$

Note that the coefficients $a_\circ^\circ$, a_1° and a_2° are functions of the scalar invariants E_I, E_{II} and E_{III}. Equation (8.119) shows that the second Piola-Kirchhoff stress tensor is formally a quadratic function of the Green-Lagrange strain tensor.

Substituting E for W in equation (1.129) yields:

$$\frac{\partial E_{III}}{\partial E_{AB}} = E_{II}\delta_{AB} - E_I E_{AB} + E_{AQ}E_{QB} = E_{III}E^{-1}_{AB}\,. \tag{8.121}$$

With (8.118a), (8.118a) and (8.121) we obtain from (8.116) that

$$\begin{aligned} S_{AB} &= \frac{\partial \psi}{\partial E_I}\delta_{AB} + \frac{\partial \psi}{\partial E_{II}}\left(E_I\delta_{AB} - E_{AB}\right) + \frac{\partial \psi}{\partial E_{III}}E_{III}E^{-1}_{AB} = \\ &= \left(\frac{\partial \psi}{\partial E_I} + E_I\frac{\partial \psi}{\partial E_{II}}\right)\delta_{AB} - \frac{\partial \psi}{\partial E_{II}}E_{AB} + E_{III}\frac{\partial \psi}{\partial E_{III}}E^{-1}_{AB} \end{aligned} \tag{8.122}$$

or

$$\boxed{S_{AB} = b^\circ_0\delta_{AB} + b^\circ_1 E_{AB} + b^\circ_2 E^{-1}_{AB}\,; \qquad \boldsymbol{S} = b^\circ_0\boldsymbol{1} + b^\circ_1\boldsymbol{E} + b^\circ_2\boldsymbol{E}^{-1}} \tag{8.123}$$

where

$$b^\circ_0 = \frac{\partial \psi}{\partial E_I} + E_I\frac{\partial \psi}{\partial E_{II}}\,, \quad b^\circ_1 = -\frac{\partial \psi}{\partial E_{II}}, \quad b^\circ_2 = \frac{\partial \psi}{\partial E_{III}}E_{III}. \tag{8.124}$$

REMARK 8.21: If the homogeneous and isotropic body is linearly elastic the constitutive equation should be a linear function of the Green-Lagrange strain tensor. Hence

$$a^\circ_2 = \frac{\partial \psi}{\partial E_{III}} = 0\,. \tag{8.125}$$

Then

$$S_{AB} = \left(\frac{\partial \psi}{\partial E_I} + E_I\frac{\partial \psi}{\partial E_{II}}\right)\delta_{AB} - \frac{\partial \psi}{\partial E_{II}}E_{AB}\,. \tag{8.126}$$

Since the right side is a linear function of the Green-Lagrange strain tensor it follows that

$$-\frac{\partial \psi}{\partial E_{II}} = 2\alpha_1 = \text{constant} \tag{8.127a}$$

and

$$\frac{\partial \psi}{\partial E_I} = \alpha_\circ E_I; \qquad \alpha_\circ = \text{constant}\,. \tag{8.127b}$$

This means that the constitutive equation assumes the form

$$S_{AB} = (\alpha_\circ - 2\alpha_1)\,E_I\delta_{AB} + 2\alpha_1 E_{AB}\,. \tag{8.128}$$

By introducing the notations

$$\lambda^\circ = \lambda = \alpha_\circ - 2\alpha_1\,, \qquad \mu^\circ = \mu = \alpha_1 \tag{8.129}$$

the constitutive equation (8.128) can be rewritten into the following form:

$$\boxed{S_{AB} = 2\mu^\circ E_{AB} + \lambda^\circ E_I\delta_{AB} = C_{ABKL}\,E_{KL}\,,} \tag{8.130}$$

where

$$C_{ABKL} = 2\mu\delta_{ABKL} + \lambda\delta_{AB}\delta_{KL} \tag{8.131}$$

is an isotropic tensor of order four. Equation (8.131) coincides with equation (8.100a) established for linear thermoelastic material behavior and presented in direct notation. The constants λ and μ are again the Lamé constants.

Remark 8.22: Based on all that has been said above, a consistent two plus one parametric constitutive equation could be the following relation:

$$S_{AB} = 2\mu^{\circ} E_{AB} + \lambda^{\circ} E_I \delta_{AB} + a_2^{o} E_{AQ} E_{QB} \tag{8.132}$$

where it is assumed that

$$a_2^{\circ} = \frac{\partial \psi}{\partial E_{III}} = \text{constant.} \tag{8.133}$$

With equation (8.132) for the derivative $\frac{\partial S_{AB}}{\partial E_{MN}}$ in (7.105) we get

$$\frac{\partial S_{AB}}{\partial E_{MN}} = \frac{\partial}{\partial E_{MN}} \left(2\mu^{\circ} E_{AB} + \lambda^{\circ} E_I \delta_{AB} + a_2^{o} E_{AQ} E_{QB}\right),$$

where

$$\frac{\partial}{\partial E_{MN}} E_{AB} = \delta_{MA} \delta_{NB}, \qquad \frac{\partial}{\partial E_{MN}} E_I = \delta_{MN}$$

and

$$\frac{\partial}{\partial E_{MN}} E_{AQ} E_{QB} = \delta_{MN} E_{AB} + E_{AB} \delta_{MN} = 2\delta_{MN} E_{AB}.$$

Hence

$$\boxed{\frac{\partial S_{AB}}{\partial E_{MN}} = 2\mu^{\circ} \delta_{MA} \delta_{NB} + \lambda^{\circ} \delta_{MN} \delta_{AB} + 2a_2^{o} \delta_{MN} E_{AB}.} \tag{8.134}$$

Assume second that $\psi = \psi(C_I, C_{II}, C_{III}) = \rho^{\circ} e(C_I, C_{II}, C_{III})$ in equations (6.67), $(8.110)_{1,2}$. Since

$$\partial E_{AB} = \frac{1}{2} \partial C_{AB}$$

equation (8.116) yields

$$S_{AB} = \frac{\partial \psi}{\partial E_{AB}} = 2 \frac{\partial \psi}{\partial C_{AB}} = \\ = 2 \frac{\partial \psi}{\partial C_I} \frac{\partial C_I}{\partial C_{AB}} + 2 \frac{\partial \psi}{\partial C_{II}} \frac{\partial C_{II}}{\partial C_{AB}} + 2 \frac{\partial \psi}{\partial E_{III}} \frac{\partial E_{III}}{\partial E_{AB}}. \tag{8.135}$$

where on the basis of equations (8.118) and (8.121) we have

$$\frac{\partial C_I}{\partial C_{AB}} = \delta_{AB}, \qquad \frac{\partial C_{II}}{\partial C_{AB}} = C_I \delta_{AB} - C_{AB} \tag{8.136a}$$

$$\frac{\partial C_{III}}{\partial E_{AB}} = C_{II} \delta_{AB} - C_I C_{AB} + C_{AQ} C_{QB} = C_{III} C_{AB}^{-1}. \tag{8.136b}$$

Making use of these derivatives we obtain from (8.135) that

$$\boxed{S_{AB} = \hat{a}_{\circ}^{\circ} \delta_{AB} + \hat{a}_1^{\circ} C_{AB} + \hat{a}_2^{\circ} C_{AQ} C_{QB}, \qquad \boldsymbol{S} = \hat{a}_{\circ}^{\circ} \boldsymbol{1} + \hat{a}_1^{\circ} \boldsymbol{C} + \hat{a}_2^{\circ} \boldsymbol{C}^2,} \tag{8.137}$$

where

$$\hat{a}_0^{\circ} = 2 \frac{\partial \psi}{\partial C_I} + 2C_I \frac{\partial \psi}{\partial C_{II}} + 2C_{II} \frac{\partial \psi}{\partial C_{III}}, \tag{8.138a}$$

$$\hat{a}_1^{\circ} = -2 \frac{\partial \psi}{\partial C_{II}} - 2C_I \frac{\partial \psi}{\partial C_{III}}, \quad \hat{a}_2^{\circ} = 2 \frac{\partial \psi}{\partial C_{III}}. \tag{8.138b}$$

By repeating the steps that resulted in equations (8.123) and (8.124) we also have

$$\boxed{S_{AB} = \hat{b}_0^\circ \delta_{AB} + \hat{b}_1^\circ C_{AB} + \hat{b}_2^\circ C_{AB}^{-1}\,, \qquad \boldsymbol{S} = \hat{b}_0^\circ \boldsymbol{1} + \hat{b}_1^\circ \boldsymbol{C} + \hat{b}_2^\circ \boldsymbol{C}^{-1}\,,} \tag{8.139}$$

where

$$\hat{b}_0^\circ = 2\left(\frac{\partial \psi}{\partial C_I} + C_I \frac{\partial \psi}{\partial C_{II}}\right), \quad \hat{b}_1^\circ = -2\frac{\partial \psi}{\partial C_{II}}, \quad \hat{b}_2^\circ = 2\frac{\partial \psi}{\partial C_{III}} C_{III}. \tag{8.140}$$

REMARK 8.23: For equilibrium problems the following field equations should be solved in material description.

Kinematic equation:

$$E_{AB} = \frac{1}{2}\left(u_{A,B}^\circ + u_{B,A}^\circ + u_{M,A}^\circ u_{M,B}^\circ\right) \qquad X_L \in V^\circ; \tag{8.141a}$$

Constitutive equation (generalized Hooke's law):

$$S_{AB} = a_\circ^\circ \delta_{AB} + a_1^\circ E_{AB} + a_2^\circ E_{AQ} E_{QB} \qquad X_L \in V^\circ; \tag{8.141b}$$

Balance equation (equilibrium equation):

$$\left(F_{AM} S_{MB}\right)_{,B} + \rho^\circ b_A^\circ = 0 \qquad X_L \in V^\circ\,. \tag{8.141c}$$

These equations are associated, in general, with the displacement boundary condition

$$u_A^\circ = \tilde{u}_A^\circ \qquad X_L \in A_u^\circ \tag{8.142a}$$

and the traction boundary condition:

$$F_{AM} S_{MB} n_B^\circ = \tilde{t}_A^\circ \qquad X_L \in A_t^\circ\,. \tag{8.142b}$$

As regards the Cauchy stress tensor we recall equation (6.69) which says that $\boldsymbol{t} = J^{-1}\boldsymbol{F}\cdot\boldsymbol{S}\cdot\boldsymbol{F}^T$. Substituting here $\boldsymbol{S}$ from (8.139) and (8.140) yields

$$t_{k\ell} = \frac{2}{J} F_{kA}\left[\left(\frac{\partial \psi}{\partial C_I} + C_I \frac{\partial \psi}{\partial C_{II}}\right)\delta_{AB} - \left(\frac{\partial \psi}{\partial C_{II}}\right) C_{AB} + \right. \\ \left. + C_{III}\frac{\partial \psi}{\partial C_{III}} C_{AB}^{-1}\right] F_{B\ell}\,, \tag{8.143}$$

where

$$F_{kA}\delta_{AB}F_{B\ell} = F_{kA}F_{A\ell} \underset{(2.49)}{=} b_{k\ell}\,, \tag{8.144a}$$

$$F_{kA}C_{AB}F_{B\ell} \underset{(2.32)}{=} F_{kA}F_{Am}F_{mB}F_{B\ell} = b_{km}b_{m\ell}\,, \tag{8.144b}$$

$$F_{kA}C_{AB}^{-1}F_{B\ell} \underset{(2.46)}{=} F_{kA}F_{Am}^{-1}F_{mB}^{-1}F_{B\ell} = \delta_{km}\delta_{m\ell} = \delta_{k\ell} \tag{8.144c}$$

and

$$C_I = b_I,\,, \qquad C_{II} = b_{II}\,, \qquad C_{III} = C_{III} \tag{8.144d}$$

which follows from the comparison of equations $(2.62)_2$ and $(2.65)_2$. Hence it holds that

$$
\begin{aligned}
t_{k\ell} &= \frac{2}{J}\left[b_{III}\frac{\partial\psi}{\partial b_{III}}\delta_{k\ell} + \left(\frac{\partial\psi}{\partial b_I} + b_I\frac{\partial\psi}{\partial b_{II}}\right) b_{k\ell} - \left(\frac{\partial\psi}{\partial b_{II}}\right) b_{km}b_{m\ell}\right],\\
\boldsymbol{t} &= \frac{2}{J}\left[b_{III}\frac{\partial\psi}{\partial b_{III}}\boldsymbol{1} + \left(\frac{\partial\psi}{\partial b_I} + b_I\frac{\partial\psi}{\partial b_{II}}\right)\boldsymbol{b} - \left(\frac{\partial\psi}{\partial b_{II}}\right)\boldsymbol{b}^2\right].
\end{aligned}
\tag{8.145}
$$

It can also be proved that

$$
\begin{aligned}
t_{k\ell} &= \frac{2}{J}\left[\left(b_{II}\frac{\partial\psi}{\partial b_{II}} + b_{III}\frac{\partial\psi}{\partial b_{III}}\right)\delta_{k\ell} + \frac{\partial\psi}{\partial b_I}b_{k\ell} - b_{III}\frac{\partial\psi}{\partial b_{II}}b^{-1}_{k\ell}\right],\\
\boldsymbol{t} &= \frac{2}{J}\left[\left(b_{II}\frac{\partial\psi}{\partial b_{II}} + b_{III}\frac{\partial\psi}{\partial b_{III}}\right)\boldsymbol{1} + \frac{\partial\psi}{\partial b_I}\boldsymbol{b} - b_{III}\frac{\partial\psi}{\partial b_{II}}\boldsymbol{b}^{-1}\right].
\end{aligned}
\tag{8.146}
$$

The proof of this statement is left for Problem 8.5.

8.6. Problems

PROBLEM 8.1: Assume that the function ϕ is an isotropic function of the symmetric tensor $\boldsymbol{E}$. Prove that the derivative $\partial\phi/\partial\boldsymbol{E}$ is coaxial with $\boldsymbol{E}$.

PROBLEM 8.2: Derive the heat conduction equation (8.76) in spatial description.

PROBLEM 8.3: Prove that the number of independent components in $\mathcal{C}_{mnk\ell}$ is 21.

PROBLEM 8.4: Prove that the inverse of the tensor $\mathcal{C}_{mnk\ell}$ satisfies the symmetry conditions $\mathcal{S}_{mnk\ell} = \mathcal{S}_{mn\ell k} = \mathcal{S}_{nm\ell k} = \mathcal{S}_{nmk\ell}$.

PROBLEM 8.5: Verify that equation (8.146) is correct.

APPENDIX A

Some longer mathematical transformations

A.1. Rodrigues formulae

A.1.1. The tensor of finite rotation. In this subsection the Rodrigues formulae is proven [2]. Let ψ, $\psi \in (\pm\pi)$ be the finite angle of rotation. The axis of rotation n is determined by the unit vector $\mathbf{n}$, $|\mathbf{n}| = 1$. The tensor of finite rotation $\boldsymbol{R}$, which is unknown at the present moment, can be devised on the basis

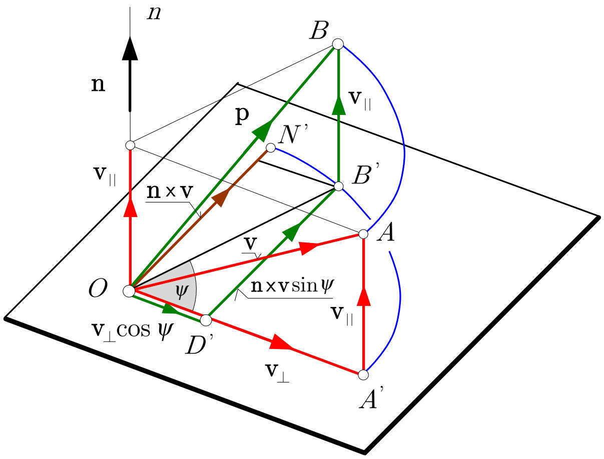

FIGURE A.1. Finite rotation

of Figure A.1. In the sequel we shall set it up going ahead step by step. The tensor $\boldsymbol{R}$ rotates (maps) the vector $\mathbf{v} = \overrightarrow{OA}$ onto the vector $\mathbf{p} = \overrightarrow{OB}$.

It is clear from Figure A.1 that

$$\mathbf{p} = \mathbf{v}_{||} + \overrightarrow{OB'} = \mathbf{v}_{||} + \overrightarrow{OD'} + \overrightarrow{D'B'}\,, \tag{A.1.1}$$

where

$$\mathbf{v}_{||} = \mathbf{n}(\mathbf{n}\cdot\mathbf{v}) = (\mathbf{n}\circ\mathbf{n})\cdot\mathbf{v}\ . \tag{A.1.2a}$$

Thus

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{||} = (\boldsymbol{1} - \mathbf{n}\circ\mathbf{n})\cdot\mathbf{v}\ . \tag{A.1.2b}$$

The leg OD' of the right angeled triangle $OD'B'$ is given by the relation

$$\overrightarrow{OD'} = \mathbf{v}_{\perp}\cos\psi = (\boldsymbol{1} - \mathbf{n}\circ\mathbf{n})\cdot\mathbf{v}\,\cos\psi\,. \tag{A.1.3}$$

Note the the vector $\overrightarrow{ON'}$ is obtained by rotating the vector $\overrightarrow{OA'} = \mathbf{v}_\perp$ about the axis n counterclockwise trough the angle $\pi/2$. Hence,

$$\overrightarrow{ON'} = \mathbf{n} \times \mathbf{v}_\perp = \underbrace{\mathbf{n} \times (\mathbf{v}_{||}}_{\text{zero vector}} + \mathbf{v}_\perp) \cdot \mathbf{v} = \mathbf{n} \times \mathbf{v} \,. \tag{A.1.4}$$

On the other hand

$$\overline{ON'} = \overline{OB'} \,,$$

by the use of which we have

$$\overline{D'B'} = \overline{OB'} \sin\psi = \overline{ON'} \sin\psi \,. \tag{A.1.5}$$

Since the vectors $\overline{ON'}$ and $\overline{D'B'}$ are parallel a comparison of equations (A.1.4) and (A.1.5) yields

$$\overrightarrow{D'B'} = \mathbf{n} \times \mathbf{v} \sin\psi \,,$$

where

$$\mathbf{n} \times \mathbf{v} = \boldsymbol{1} \cdot (\mathbf{n} \times \mathbf{v}) = (\boldsymbol{1} \times \mathbf{n}) \cdot \mathbf{v} \,.$$

Consequently,

$$\overrightarrow{D'B'} = (\boldsymbol{1} \times \mathbf{n}) \cdot \mathbf{v} \sin\psi \,. \tag{A.1.6}$$

After substituting (A.1.2a), (A.1.3) and (A.1.6) into equation (A.1.1) we get the vector $\mathbf{p}$ in the following form:

$$\begin{aligned} \mathbf{p} &= [\mathbf{n} \circ \mathbf{n} + (\boldsymbol{1} - \mathbf{n} \circ \mathbf{n}) \cos\psi + \boldsymbol{1} \times \mathbf{n} \sin\psi] \cdot \mathbf{v} \\ &= [\boldsymbol{1} \cos\psi + (1 - \cos\psi)\, \mathbf{n} \circ \mathbf{n} + \boldsymbol{1} \times \mathbf{n} \sin\psi\,] \cdot \mathbf{v} \,. \end{aligned} \tag{A.1.7}$$

Here

$$\boxed{\begin{aligned} \boldsymbol{R} &= \boldsymbol{1} \cos\psi + (1 - \cos\psi)\, \mathbf{n} \circ \mathbf{n} + \boldsymbol{1} \times \mathbf{n} \sin\psi \,, \\ R_{k\ell} &= \delta_{k\ell} \cos\psi + (1 - \cos\psi)\, n_k n_\ell + \delta_{kn} e_{nr\ell}\, n_r \sin\psi \end{aligned}} \tag{A.1.8}$$

is the tensor of finite rotation.

A.1.2. Is the tensor of finite rotation an orthogonal one. On the basis of the results obtained in the previous Subsection it is worth investigating if the tensor

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{1} \cos\psi + (Q_{III} - \cos\psi)\, \mathbf{n} \circ \mathbf{n} + \boldsymbol{1} \times \mathbf{n} \sin\psi \,, \\ Q_{k\ell} &= \delta_{k\ell} \cos\psi + (Q_{III} - \cos\psi)\, n_k n_\ell + \delta_{kn} e_{nr\ell}\, n_r \sin\psi \,, \end{aligned} \tag{A.1.9}$$

which can be regarded as if it were a generalization of the tensor $\boldsymbol{R}$ in equation (A.1.8) since Q_{III} is an unknown parameter here, is orthogonal and if yes under what conditions.

To this end we have to investigate what properties the product $\boldsymbol{Q} \cdot \boldsymbol{Q}^T$ has. It is obvious that

$$\boldsymbol{Q}^* = \boldsymbol{1} \cos\psi + (Q_{III} - \cos\psi)\, \mathbf{n} \circ \mathbf{n} \,, \qquad \boldsymbol{Q}^* = (\boldsymbol{Q}^*)^T \tag{A.1.10}$$

is the symmetric part of the tensor $\boldsymbol{Q}$. Note that

$$(\boldsymbol{1} \times \mathbf{n})^T = (\mathbf{i}_k \circ \mathbf{i}_k \times \mathbf{n})^T = (\mathbf{i}_k \times \mathbf{n}) \circ \mathbf{i}_k = -(\mathbf{n} \times \mathbf{i}_k) \circ \mathbf{i}_k = -\mathbf{n} \times \boldsymbol{1} \,. \tag{A.1.11}$$

Hence,

$$\boldsymbol{Q} = \boldsymbol{Q}^* + \boldsymbol{1} \times \mathbf{n} \sin\psi \qquad \text{and} \qquad \boldsymbol{Q}^T = \boldsymbol{Q}^* - \mathbf{n} \times \boldsymbol{1} \sin\psi \ , \tag{A.1.12}$$

by the use of which we have

$$\boldsymbol{Q}\cdot\boldsymbol{Q}^T = \boldsymbol{Q}^*\cdot\boldsymbol{Q}^* + [(\boldsymbol{1} \times \mathbf{n})\cdot\boldsymbol{Q}^* - \boldsymbol{Q}^*\cdot(\mathbf{n}\times\boldsymbol{1})]\sin\psi - \sin^2\psi\,(\boldsymbol{1}\times\mathbf{n})\cdot(\mathbf{n}\times\boldsymbol{1}) \ . \tag{A.1.13}$$

To establish the final result we shall need the following formulas which provide appropriate expressions for the various terms on the right side of equation (A.1.13). We shall present them with the necessary explanations:

(a) By utilizing definition (A.1.10) for $\boldsymbol{Q}^*$ we get

$$(\boldsymbol{1} \times \mathbf{n})\cdot\boldsymbol{Q}^* = \boldsymbol{1}\cdot(\mathbf{n}\times\boldsymbol{Q}^*) = \mathbf{n}\times\boldsymbol{Q}^* = \mathbf{n}\times\boldsymbol{1}\cos\psi \ . \tag{A.1.14a}$$

(b) It can be obtained in the same way that

$$\boldsymbol{Q}^*\cdot(\mathbf{n}\times\boldsymbol{1}) = (\boldsymbol{Q}^*\times\mathbf{n})\cdot\boldsymbol{1} = \boldsymbol{Q}^*\times\mathbf{n} = \boldsymbol{1}\times\mathbf{n}\cos\psi \ . \tag{A.1.14b}$$

(c) It is not too difficult to verify that the products $\mathbf{n}\times\boldsymbol{1}$ and $\boldsymbol{1}\times\mathbf{n}$ in the previous two equations are the same:

$$\begin{aligned}\mathbf{n}\times\boldsymbol{1} &= n_r\,\mathbf{i}_r\times\mathbf{i}_k\circ\mathbf{i}_k = n_r e_{rks}\,\mathbf{i}_s\circ\mathbf{i}_k = \\ &= \mathbf{i}_k\circ\mathbf{i}_s\,e_{krs}\,n_r = \mathbf{i}_k\circ\mathbf{i}_k\times\mathbf{i}_r\,n_r = \boldsymbol{1}\times\mathbf{n} \ .\end{aligned} \tag{A.1.14c}$$

(d) A further partial result can be obtained by the following transformation:

$$\begin{aligned}(\boldsymbol{1}\times\mathbf{n})\cdot(\mathbf{n}\times\boldsymbol{1}) &= (\mathbf{i}_k\circ\mathbf{i}_k\times\mathbf{n})\cdot(\mathbf{n}\times\mathbf{i}_\ell\circ\mathbf{i}_\ell) = \\ &= \underbrace{(\mathbf{i}_k\times\mathbf{n})\cdot(\mathbf{n}\times\mathbf{i}_\ell)}_{\mathbf{i}_k\cdot[\mathbf{n}\times(\mathbf{n}\times\mathbf{i}_\ell)]}\,\mathbf{i}_k\circ\mathbf{i}_\ell = [\mathbf{i}_k\cdot(n_\ell\mathbf{n}-\mathbf{i}_\ell)]\,\mathbf{i}_k\circ\mathbf{i}_\ell = \\ &= (n_k n_\ell - \delta_{k\ell})\,\mathbf{i}_k\circ\mathbf{i}_\ell = \mathbf{n}\circ\mathbf{n} - \boldsymbol{1} \ .\end{aligned} \tag{A.1.14d}$$

(e) On the basis of equation (A.1.10), which defines $\boldsymbol{Q}^*$, we have

$$\begin{aligned}\boldsymbol{Q}^*\cdot(\boldsymbol{Q}^*)^T &= \boldsymbol{Q}^*\cdot\boldsymbol{Q}^* = \\ &= \boldsymbol{1}\cos^2\psi + 2\cos\psi\,(Q_{III}-\cos\psi)\,\mathbf{n}\circ\mathbf{n} + (Q_{III}-\cos\psi)^2\,\mathbf{n}\circ\mathbf{n} = \\ &= \boldsymbol{1}\cos^2\psi + \left(Q_{III}^2-\cos^2\psi\right)\mathbf{n}\circ\mathbf{n} \ .\end{aligned} \tag{A.1.14e}$$

Substitution of the above partial results (A.1.14a),...,(A.1.14e) into equation (A.1.13) set up for the product $\boldsymbol{Q}\cdot\boldsymbol{Q}^T$ yields

$$\begin{aligned}\boldsymbol{Q}\cdot\boldsymbol{Q}^T &= \boldsymbol{1}\cos^2\psi + \left(Q_{III}^2-\cos^2\psi\right)\mathbf{n}\circ\mathbf{n} + \boldsymbol{1}\sin^2\psi - \mathbf{n}\circ\mathbf{n}\sin^2\psi \\ &= \boldsymbol{1} + \left(Q_{III}^2-1\right)\mathbf{n}\circ\mathbf{n} \ .\end{aligned} \tag{A.1.15}$$

The tensor $\boldsymbol{Q}$ defined by equation (A.1.9) is orthogonal if and only if $\boldsymbol{Q}\cdot\boldsymbol{Q}^T = \boldsymbol{1}$. It follows from equation (A.1.15) at once that this condition is always satisfied if

$$Q_{III}^2 = 1 \ , \qquad \text{or} \qquad Q_{III} = \pm 1 \ .$$

In the sequal we shall clarify the meaning of Q_{III}. Assume that the coordinate axis x_1 coincides with the axis n. Then

$$\mathbf{n} = \mathbf{i}_1$$

and

$$\det(\boldsymbol{Q}\cdot\boldsymbol{Q}^T) = \det\big(\boldsymbol{1} + \left(Q_{III}^2 - 1\right)\,\mathbf{i}_1 \circ \mathbf{i}_1\big) =$$
$$= \begin{vmatrix} 1 + Q_{III}^2 - 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = Q_{III}^2\,.$$

which shows that Q_{III} is the third scalar invariant of the tensor $\boldsymbol{Q}$.

To sum up the results it is proven that the tensor $\boldsymbol{Q}$ defined by equation (A.1.9) is orthogonal if $Q_{III} = \pm 1$, where Q_{III} is the third scalar invariant of the tensor. For $Q_{III} = 1$ the tensor $\boldsymbol{Q}$ coincides with the tensor of finite rotation $\boldsymbol{R}$. Consequently, then the tensor $\boldsymbol{Q}$ is proper orthogonal.

Let us assume that the tensor $\boldsymbol{Q}$ is known. Multiply equation (A.1.9) by $\delta_{k\ell}$. Since the Kronecker delta is an index renaming operator we get

$$\delta_{k\ell}Q_{k\ell} = Q_I = 3\cos\psi + (Q_{III} - \cos\psi) + e_{\ell r \ell}\,n_r \sin\psi\,,$$

where the last term is clearly zero. Hence the angle of rotation can be calculated form the following equation:

$$\boxed{\cos\psi = \frac{1}{2}\left(Q_I - Q_{III}\right)\,.} \tag{A.1.16}$$

Multiply now equation (A.1.9) by $e_{k\ell m}$ and take into account that the double dot product of a skew and symmetric tensor is zero. We obtain

$$e_{k\ell m}Q_{k\ell} = \delta_{kn}e_{k\ell m}e_{n\ell r}n_r \sin\psi = -e_{n\ell m}e_{n\ell r}n_r \sin\psi = -2\delta_{mr}n_r = -2n_m\,.$$

Hence, the unit vector of the rotation axis is given by the equation

$$\boxed{n_m = -\frac{e_{k\ell m}Q_{k\ell}}{\sin\psi}\,.} \tag{A.1.17}$$

It is customary to give the tensor of finite rotation $R_{k\ell}$ in terms of the rotation vector

$$\boldsymbol{\psi} = \psi_k\,\mathbf{i}_k = \psi\,\mathbf{n} \tag{A.1.18}$$

or of the corresponding skew rotation tensor

$$\Psi_{k\ell} = -e_{k\ell m}\psi_m\,. \tag{A.1.19}$$

Substituting the rotation vector (A.1.16) into (A.1.8) results in

$$\boxed{R_{k\ell} = \delta_{k\ell}\cos\psi + \frac{1-\cos\psi}{\psi^2}\,\psi_k\psi_l - \frac{\sin\psi}{\psi}\,e_{k\ell m}\,\psi_m,} \tag{A.1.20}$$

which is the second form of the tensor of finite rotation.

Consider now the transformation

$$\varPsi_{ks}\varPsi_{s\ell} = e_{ksm}\,\psi_m\;e_{s\ell r}\,\psi_r = e_{s\ell r}\,e_{smk}\;\psi_r\,\psi_m =$$
$$= (\delta_{\ell m}\,\delta_{rk} - \delta_{rm}\,\delta^{\ell k})\,\psi_r\,\psi_m = \psi_k\,\psi_\ell - \delta_{k\ell}\,\psi_r\,\psi_r\ ,$$

from where

$$\psi_k\,\psi_\ell = \varPsi_{ks}\varPsi_{s\ell} + \delta_{k\ell}\,\psi_r\,\psi_r\ .$$

Making use of this result we may rewrite equation (A.1.20):

$$R_{k\ell} = \delta_{k\ell}\,\cos\psi + \frac{1-\cos\psi}{\psi^2}\,(\varPsi_{ks}\varPsi_{s\ell} + \delta_{k\ell}\,\psi_r\,\psi_r) - \frac{\sin\psi}{\psi}\,\varepsilon_{klm}\,\psi^m.$$

Hence

$$\boxed{R_{k\ell} = \delta_{kl} + \frac{1-\cos\psi}{\psi^2}\,\varPsi_{ks}\,\varPsi_{s\ell} + \frac{\sin\psi}{\psi}\,\varPsi_{k\ell}\,,} \tag{A.1.21}$$

which is the third form of the tensor of finite rotation.

A.2. Isotropic tensor functions

A.2.1. Isotropic tensors. Subsection 1.4.7 defines the concept of isotropic tensors. It is shown that (a) a scalar is isotropic, (b) there are no isotropic vectors, (c) the only isotropic tensor is of the form $\alpha\delta_{k\ell}$, (d) the isotropic triads are of the form $\alpha e_{k\ell r}$ and finally that (e) the tetrads defined by equations (1.168b), (1.168c) are also isotropic. These tetrads play, later, an important role in the theory of constitutive equations.

A.2.2. Real istotropic scalar functions. Let $\boldsymbol{E}' = E'_{k\ell}\,\mathbf{i}'_k \circ \mathbf{i}'_\ell$ be a symmetric tensor. The eigenvalues and eigenvectors of $\boldsymbol{E}'$ are denoted by λ'_r and $\mathbf{n}'_r$ ($|\mathbf{n}'_r| = 1$), respectively. Assume further that $\boldsymbol{\mathcal{T}} = \mathcal{T}_{k\ell'}\,\mathbf{i}_k \circ \mathbf{i}'_\ell$ is the transformation tensor between the primed and unprimed reference frames (between the two observers). The eigenvalues and eigenvectors of the symmetric tensor $\boldsymbol{E} = \boldsymbol{\mathcal{T}}\cdot\boldsymbol{E}'\cdot\boldsymbol{\mathcal{T}}^T$ are denoted by λ_r and $\mathbf{n}_r$. It follows from equations (1.112) and (1.113) that the characteristic equations of the tensors $\boldsymbol{E}'$ and $\boldsymbol{E} = \boldsymbol{\mathcal{T}}\cdot\boldsymbol{E}'\cdot\boldsymbol{\mathcal{T}}^T$ are of the form

$$P_3(\lambda') = -|E'_{k\ell} - \lambda'\delta_{k\ell}| = (\lambda')^3 - E'_I(\lambda')^2 + E'_{II}\lambda' - E'_{III} = 0 \tag{A.2.22a}$$

and

$$P_3(\lambda) = -|E_{rs} - \lambda\delta_{rs}| = \lambda^3 - E_I\lambda^2 + E_{II}\lambda - E_{III} =$$
$$= -|\mathcal{T}_{rk}\,(E'_{k\ell} - \lambda\delta_{k\ell})\,\mathcal{T}_{\ell s}| = |\mathcal{T}_{rk}|\,|E'_{k\ell} - \lambda\delta_{k\ell}|\,|\mathcal{T}_{\ell s}| \underset{|\mathcal{T}_{rk}|=|\mathcal{T}_{\ell s}|=1}{=\!\!\uparrow\!\!=}$$
$$= |E'_{k\ell} - \lambda\delta_{k\ell}| = \lambda^3 - E'_I\lambda^2 + E'_{II}\lambda - E'_{III} = 0\,. \tag{A.2.22b}$$

A comparison of equations (A.2.22) shows that the characteristic equations of the tensors $\boldsymbol{E}'$ and $\boldsymbol{E} = \boldsymbol{\mathcal{T}}\cdot\boldsymbol{E}'\cdot\boldsymbol{\mathcal{T}}^T$ are the same. Hence, the eigenvalues and scalar invariants coincide with each other:

$$\begin{gathered}\lambda'_r = \lambda_r,\\ E'_I = E_I\,,\qquad E'_{II} = E_{II}\,,\qquad E'_{III} = E_{III}\,.\end{gathered} \tag{A.2.23}$$

It also holds for any $\mathcal{T}_{rk}$ that

$$(E_{rs} - \lambda\delta_{rs})\, n_s = \underbrace{\mathcal{T}_{rk} E'_{k\ell} \mathcal{T}_{\ell s}}_{E_{rs}} n_s - \lambda \underbrace{\mathcal{T}_{rk} \mathcal{T}_{ks}}_{\delta_{rs}} n_s = \\ = \mathcal{T}_{rk}\left(E'_{k\ell} - \lambda' \delta_{k\ell}\right) \underbrace{\mathcal{T}_{\ell s} n_s}_{n'_\ell} = 0 \qquad \text{(A.2.24)}$$

which means that the eigenvectors are also the same: $n'_\ell = \mathcal{T}_{\ell s} n_s$.

The inverse statement also holds. Let now $\boldsymbol{E}'$ and $\boldsymbol{E}$ and be two symmetric tensors regarded in the primed and unprimed reference frames for which the scalar invariants and the eigenvalues are the same. Then there exists such a transformation tensor which satisfies the relation

$$\boldsymbol{E} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{E}' \cdot \boldsymbol{\mathcal{T}}^T . \qquad \text{(A.2.25)}$$

Assume that $\boldsymbol{\mathcal{T}} = \mathcal{T}_{k\ell'}\ \mathbf{i}_k \circ \mathbf{i}'_\ell$. It is obvious that this tensor really satisfies equation (A.2.25).

Remark A.1: It follows from the previous line of thought that the eigenvalues and eigenvectors of the tensors $\boldsymbol{E}'$ and $\boldsymbol{E}$ are the same. This conclusion reflects the fulfillment of the natural requirement that the eigenvalues and principal directions of symmetric tensors should be independent of the observers.

Let $\boldsymbol{E}'$ be a symmetric tensor: $\boldsymbol{E}' = \boldsymbol{E}'^T$. The scalar valued tensor function $f = f(\boldsymbol{E}') = f(E'_{k\ell})$ is said to be a real isotropic tensor function if it holds for all possible transformation tensors $\boldsymbol{\mathcal{T}}$ that

$$f(\boldsymbol{E}') = f(\boldsymbol{\mathcal{T}} * \boldsymbol{E}') = f(\boldsymbol{\mathcal{T}} \cdot \boldsymbol{E}' \cdot \boldsymbol{\mathcal{T}}^T) = f(\boldsymbol{E}) . \qquad \text{(A.2.26)}$$

Remark A.2: The expression for all possible transformation tensors $\boldsymbol{\mathcal{T}}$ reflects the fact that equation (A.2.26) should be valid in any of the possible coordinate systems the observers select for their reference frames.

Remark A.3: Requirement (A.2.26) says that the value of the scalar f should be independent of the observers: it must be the same for the two observers.

It follows from equations (A.2.22), (A.2.23) and (A.2.25) that a real isotropic tensor function should be independent of $\boldsymbol{\mathcal{T}}$; therefore it can be represented as a function of the three scalar invariants $E'_I = E_I$, $E'_{II} = E_{II}$ and $E'_{III} = E_{III}$ only:

$$\boxed{f(\boldsymbol{E}') = f(\boldsymbol{\mathcal{T}} \cdot \boldsymbol{E}' \cdot \boldsymbol{\mathcal{T}}^T) = f(E'_I, E'_{II}, E'_{III}) = f(E_I, E_{II}, E_{III}) .} \qquad \text{(A.2.27)}$$

According to equation (1.114) the scalar invariants can be given in terms of the eigenvalues:

$$E_I = \lambda_1 + \lambda_2 + \lambda_3 , \quad E_{II} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 , \quad E_{III} = \lambda_1\lambda_2\lambda_3 . \qquad \text{(A.2.28)}$$

Consequently,

$$f(E_I, E_{II}, E_{III}) = \check{f}(\lambda_1, \lambda_2, \lambda_3) . \qquad \text{(A.2.29)}$$

If λ_2, λ_3 and λ_1 are substituted for λ_1, λ_2 and λ_3 in (A.2.28) the scalar invariants remain unchanged. This observation leads to the cyclic interchangeability of the arguments λ_ℓ in representation (A.2.29) of the real isotropic scalar functions:

$$\boxed{\check{f}(\lambda_1,\lambda_2,\lambda_3)=\check{f}(\lambda_2,\lambda_3,\lambda_1)=\check{f}(\lambda_3,\lambda_1,\lambda_2)\,.} \tag{A.2.30}$$

A.2.3. Isotropic tensor valued tensor functions. The tensor valued tensor function (or tensor-tensor function)

$$\boldsymbol{\Phi}=\boldsymbol{\Phi}(\boldsymbol{E}');\qquad \boldsymbol{\Phi}=\boldsymbol{\Phi}^T\,,\quad \boldsymbol{E}'=\boldsymbol{E}'^T \tag{A.2.31}$$

is said to be isotropic if

$$\boldsymbol{\mathcal{T}}\cdot\boldsymbol{\Phi}(\boldsymbol{E}')\cdot\boldsymbol{\mathcal{T}}^T=\boldsymbol{\Phi}(\boldsymbol{\mathcal{T}}*\boldsymbol{E}')=\boldsymbol{\Phi}(\boldsymbol{\mathcal{T}}\cdot\boldsymbol{E}'\cdot\boldsymbol{\mathcal{T}}^T)=\boldsymbol{\Phi}(\boldsymbol{E}) \tag{A.2.32}$$

holds for all transformation tensors $\boldsymbol{\mathcal{T}}$.

Remark A.4: Assume that the symmetric tensor $\boldsymbol{\Phi}$ – the tensor-tensor function $\boldsymbol{\Phi}$ – describes a physical quantity. If requirement (A.2.32) is fulfilled then the tensor $\boldsymbol{\Phi}$ is independent of the observers: it is the same for the two observers.

Let ϕ_0, $\phi_1,\dots,\phi_n$ $(k\geq 0)$ be real isotropic scalar functions for which it holds that: $\phi_\ell=\phi_\ell(\boldsymbol{E}')=\phi_\ell(E'_I,E'_{II},E'_{III})$ $(\ell=1,2,\dots,k)$. Then the tensor polynomial

$$\boldsymbol{\Phi}(\boldsymbol{E}')=\phi_0\boldsymbol{1}+\phi_1\boldsymbol{E}'+\phi_2(\boldsymbol{E}')^2+\phi_3(\boldsymbol{E}')^3+\cdots+\phi_k(\boldsymbol{E}')^k \tag{A.2.33}$$

is an isotropic tensor valued tensor function. Instead of giving a detailed proof we shall consider the case when $\boldsymbol{\Phi}(\boldsymbol{E}')=\phi_2(\boldsymbol{E}')^2$. Then

$$\boldsymbol{\mathcal{T}}\cdot\boldsymbol{\Phi}(\boldsymbol{E}')\cdot\boldsymbol{\mathcal{T}}^T=\phi_2\,\boldsymbol{\mathcal{T}}\cdot(\boldsymbol{E}')^2\cdot\boldsymbol{\mathcal{T}}^T \tag{A.2.34a}$$

and

$$\boldsymbol{\Phi}(\boldsymbol{\mathcal{T}}\cdot\boldsymbol{E}'\cdot\boldsymbol{\mathcal{T}}^T)=\phi_2\,\boldsymbol{\mathcal{T}}\cdot\boldsymbol{E}'\cdot\underbrace{\boldsymbol{\mathcal{T}}^T\cdot\boldsymbol{\mathcal{T}}}_{\boldsymbol{1}}\cdot\boldsymbol{E}'\cdot\boldsymbol{\mathcal{T}}^T=\phi_2\,\boldsymbol{\mathcal{T}}\cdot(\boldsymbol{E}')^2\cdot\boldsymbol{\mathcal{T}}^T. \tag{A.2.34b}$$

Since the right sides of (A.2.34a) and (A.2.34b) are equal and ϕ_2 is isotropic it follows that $\phi_2(\boldsymbol{E}')^2$ is also isotropic. The line of thought is similar concerning the other terms on the right side of (A.2.33).

Remark A.5: It follows from the Cayley-Hamilton (1.128) theorem (1.128) that

$$(\boldsymbol{E}')^3=E'_I(\boldsymbol{E}')^2-E'_{II}\boldsymbol{E}'+E'_{III}\boldsymbol{1}\,.$$

Substituting this equation repeatedly into the right side of (A.2.33) yields an isotropic quadratic tensor polynomial in $\boldsymbol{E}$:

$$\boldsymbol{\Phi}(\boldsymbol{E}')=\phi_0\boldsymbol{1}+\phi_1\boldsymbol{E}'+\phi_2(\boldsymbol{E}')^2+\phi_3(\boldsymbol{E}')^3+\cdots+\phi_k(\boldsymbol{E}')^k=$$
$$=\psi_0\boldsymbol{1}+\psi_1\boldsymbol{E}'+\psi_2(\boldsymbol{E}')^2 \tag{A.2.35}$$

in which ψ_0, ψ_1 and ψ_2 are isotropic scalars.

REMARK A.6: The tensors $\boldsymbol{E}$ ($\boldsymbol{E} = \boldsymbol{E}^T$) and $\boldsymbol{E}^k$ ($k \geq 2$) are coaxial. If $k = 2$ we may write

$$\boldsymbol{E}^2 \cdot \mathbf{n}_s = \boldsymbol{E} \cdot \boldsymbol{E} \cdot \mathbf{n}_s = \lambda_s \cdot \boldsymbol{E} \cdot \mathbf{n}_s = \lambda_s^2 \mathbf{n}_s$$

which shows that $\boldsymbol{E}$ and $\boldsymbol{E}^2$ are really coaxial. For powers higher than two the proof is similar: $\boldsymbol{E}^k \cdot \mathbf{n}_s = \lambda_s^k \mathbf{n}_s$.

It is now clear that the eigenvectors of the isotropic tensor valued function given by the tensor polynomial (A.2.33) are the same as those of the tensor $\boldsymbol{E}'$.

We shall prove that this statement is valid for any isotropic tensor valued tensor function $\boldsymbol{\Phi}(\boldsymbol{E}')$ [41, 80, 85].

For the observer in the primed coordinate system

$$\boldsymbol{E}' = \sum_{\ell=1}^{3} \lambda'_\ell \, \mathbf{n}'_\ell \circ \mathbf{n}'_\ell \tag{A.2.36}$$

is the spectral decomposition of the tensor $\boldsymbol{E}$. For the observer in the unprimed coordinate system it holds that

$$\boldsymbol{E} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{E}' \cdot \boldsymbol{\mathcal{T}}^T = \sum_{\ell=1}^{3} \lambda'_\ell \, \underbrace{\boldsymbol{\mathcal{T}} \cdot \mathbf{n}'_\ell}_{\mathbf{n}_\ell} \circ \underbrace{\mathbf{n}'_\ell \cdot \boldsymbol{\mathcal{T}}^T}_{\mathbf{n}_\ell} = \sum_{\ell=1}^{3} \lambda_\ell \, \mathbf{n}_\ell \circ \mathbf{n}_\ell \,, \qquad \lambda'_\ell = \lambda_\ell \,. \tag{A.2.37}$$

Assume that the unit vectors for the primed and unprimed coordinate systems used by the two observers are given by the following equations:

$$\mathbf{i}'_1 = \mathbf{n}'_1 \,, \qquad \mathbf{i}'_2 = \mathbf{n}'_2 \,, \qquad \mathbf{i}'_3 = \mathbf{n}'_3 \tag{A.2.38a}$$

and

$$\mathbf{i}_1 = \mathbf{n}_1 \,, \qquad \mathbf{i}_2 = -\mathbf{n}_2 \,, \qquad \mathbf{i}_3 = -\mathbf{n}_3 \,. \tag{A.2.38b}$$

Then

$$\boldsymbol{\mathcal{T}} = \mathbf{n}_1 \circ \mathbf{n}'_1 - \mathbf{n}_2 \circ \mathbf{n}'_2 - \mathbf{n}_3 \circ \mathbf{n}'_3 \tag{A.2.39}$$

is the transformation tensor. It is obvious that

$$\boldsymbol{\mathcal{T}} \cdot \mathbf{n}'_1 = \mathbf{n}_1 \,, \qquad \boldsymbol{\mathcal{T}} \cdot \mathbf{n}'_2 = -\mathbf{n}_2 \,, \qquad \boldsymbol{\mathcal{T}} \cdot \mathbf{n}'_3 = -\mathbf{n}_3 \,. \tag{A.2.40}$$

Consequently,

$$\boldsymbol{\mathcal{T}} \cdot \boldsymbol{E}' \cdot \boldsymbol{\mathcal{T}}^T = \underset{(A.2.39)\ (A.2.36)}{\uparrow} =$$

$$= (\mathbf{n}_1 \circ \mathbf{n}'_1 - \mathbf{n}_2 \circ \mathbf{n}'_2 - \mathbf{n}_3 \circ \mathbf{n}'_3) \cdot \left(\sum_{\ell=1}^{3} \lambda'_\ell \, \mathbf{n}'_\ell \circ \mathbf{n}'_\ell \right) \cdot (\mathbf{n}'_1 \circ \mathbf{n}_1 - \mathbf{n}'_2 \circ \mathbf{n}_2 - \mathbf{n}'_3 \circ \mathbf{n}_3) =$$

$$= \sum_{\ell=1}^{3} \lambda'_\ell \, \mathbf{n}_\ell \circ \mathbf{n}_\ell = \underset{\mathbf{n}_\ell \circ \mathbf{n}_\ell = \mathbf{n}'_\ell \circ \mathbf{n}'_\ell}{\uparrow} = \sum_{\ell=1}^{3} \lambda'_\ell \, \mathbf{n}'_\ell \circ \mathbf{n}'_\ell = \boldsymbol{E}',$$

where it has been taken into account that $\mathbf{n}_\ell \circ \mathbf{n}_\ell = \mathbf{n}'_\ell \circ \mathbf{n}'_\ell$, which follows from the selection of the two coordinate systems for the observers – see (A.2.38).

Let $\boldsymbol{\Phi}(\boldsymbol{E}')$ be an isotropic tensor valued tensor function. With regard to the previous equation we have

$$\boldsymbol{\mathcal{T}} \cdot \boldsymbol{\Phi}(\boldsymbol{E}') \cdot \boldsymbol{\mathcal{T}}^T = \boldsymbol{\Phi}(\boldsymbol{\mathcal{T}} \cdot \boldsymbol{E}' \cdot \boldsymbol{\mathcal{T}}^T) = \boldsymbol{\Phi}(\boldsymbol{E}')$$

Dot multiplying this equation from right by $\boldsymbol{\mathcal{T}}$ yields

$$\boldsymbol{\Phi}(\boldsymbol{E}') \cdot \boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{\Phi}(\boldsymbol{E}') \, .$$

It also holds that

$$\boldsymbol{\Phi}(\boldsymbol{E}') \cdot \boldsymbol{\mathcal{T}} \cdot \mathbf{n}_1' = \boldsymbol{\Phi}(\boldsymbol{E}') \cdot \mathbf{n}_1' = \boldsymbol{\mathcal{T}} \cdot \boldsymbol{\Phi}(\boldsymbol{E}') \cdot \mathbf{n}_1'$$

from where we get

$$\boldsymbol{\mathcal{T}} \cdot \boldsymbol{\Phi}(\boldsymbol{E}') \cdot \mathbf{n}_1' = \boldsymbol{\Phi}(\boldsymbol{E}') \cdot \mathbf{n}_1' \, .$$

Since the only vector the tensor $\boldsymbol{\mathcal{T}}$ maps onto itself is $\mathbf{n}_1'$ it follows that the vector $\boldsymbol{\Phi}(\boldsymbol{E}') \cdot \mathbf{n}_1'$ is parallel to $\mathbf{n}_1'$. In other words: the principal direction $\mathbf{n}_1'$ of the tensor $\boldsymbol{E}'$ is that of the tensor valued tensor function $\boldsymbol{\Phi}(\boldsymbol{E}')$. If we renumber the principal directions cyclically we can come to the conclusion on the basis of the previous line of thought that the other two principal directions of the tensor $\boldsymbol{E}'$ are also principal directions of the tensor valued tensor function $\boldsymbol{\Phi}(\boldsymbol{E}')$. Consequently, the tensor $\boldsymbol{\Phi}(\boldsymbol{E}')$ and the tensor valued tensor function $\boldsymbol{\Phi}(\boldsymbol{E}')$ are coaxial. Thus

$$\boldsymbol{\Phi}(\boldsymbol{E}') \cdot \mathbf{n}_\ell' = \mu_\ell' \mathbf{n}_\ell' \, , \qquad \mu_\ell' = \mathbf{n}_\ell' \cdot \boldsymbol{\Phi}(\boldsymbol{E}') \cdot \mathbf{n}_\ell' \, , \quad \text{(no summ on } \ell) \tag{A.2.41}$$

where μ_ℓ' is an eigenvalue of the tensor $\boldsymbol{\Phi}(\boldsymbol{E}')$. Its spectral decomposition assumes the form

$$\boldsymbol{\Phi}(\boldsymbol{E}') = \sum_{\ell=1}^{3} \mu_\ell' \, \mathbf{n}_\ell' \circ \mathbf{n}_\ell' \, . \tag{A.2.42}$$

In order to clarify what mathematical form the tensor valued tensor function $\boldsymbol{\Phi}(\boldsymbol{E}')$ may have we shall consider three separate case:

(a) Assume that the eigenvalues of the tensor $\boldsymbol{E}'$ are different: $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Assume further that

$$\underbrace{\sum_{\ell=1}^{3} \mu_\ell' \, \mathbf{n}_\ell' \circ \mathbf{n}_\ell'}_{\boldsymbol{\Phi}(\boldsymbol{E}')} = \beta_o \underbrace{\sum_{\ell=1}^{3} \mathbf{n}_\ell' \circ \mathbf{n}_\ell'}_{\boldsymbol{1}} + \beta_1 \underbrace{\sum_{\ell=1}^{3} \lambda_\ell' \, \mathbf{n}_\ell' \circ \mathbf{n}_\ell'}_{\boldsymbol{E}'} + \beta_2 \underbrace{\sum_{\ell=1}^{3} (\lambda_\ell')^2 \, \mathbf{n}_\ell' \circ \mathbf{n}_\ell'}_{(\boldsymbol{E}')^2} \tag{A.2.43}$$

where β_o, β_1 and β_2 are unknown scalars. Note that equation (A.2.43) is equivalent to the following three scalar equations:

$$\begin{aligned} \mu_1 &= \beta_o + \beta_1 \lambda_1' + \beta_2 (\lambda_1')^2 \, , \\ \mu_2 &= \beta_o + \beta_1 \lambda_2' + \beta_2 (\lambda_2')^2 \, , \\ \mu_3 &= \beta_o + \beta_1 \lambda_3' + \beta_2 (\lambda_3')^2 \, . \end{aligned} \tag{A.2.44}$$

This is a linear equation system for the unknown β_o, β_1 and β_2. It can be checked with ease that the system determinant is not zero:

$$d = (\lambda'_1 - \lambda'_2)(\lambda'_2 - \lambda'_3)(\lambda'_3 - \lambda'_1) \neq 0 \,.$$

Hence there exist a unique solution for the three unknowns. Consequently, it follows from (A.2.43) that the tensor valued tensor function $\boldsymbol{\Phi}(\boldsymbol{E}')$ can be given in the form of a quadratic polynomial of $\boldsymbol{E}'$ if the eigenvalues of $\boldsymbol{E}'$ are different:

$$\boldsymbol{\Phi}(\boldsymbol{E}') = \beta_o \boldsymbol{1} + \beta_1 \boldsymbol{E}' + \beta_2 (\boldsymbol{E}')^2 \,. \tag{A.2.45}$$

It follows from (A.2.44) that μ_ℓ depends on λ_1, λ_2 and λ_3 only. Consequently the coefficients β_o, β_1 and β_3 are also functions of λ_1, λ_2 and λ_3. Since the eigenvalues λ_ℓ are the functions of the scalar invariants E'_I, E'_{II} and E'_{III} we can come to the conclusion that

$$\beta_o = \beta_o(E'_I, E'_{II}, E'_{III}) \,, \quad \beta_1 = \beta_1(E'_I, E'_{II}, E'_{III}) \,,$$
$$\beta_2 = \beta_2(E'_I, E'_{II}, E'_{III})$$

which shows that the scalars β_o, β_1 and β_3 are isotropic functions. In a view of this fact a comparison of (A.2.45) and (A.2.35) proves that representation (A.2.35) is really isotropic.

(b) Assume now that two eigenvalues of the tensor $\boldsymbol{E}'$ coincide with each other: $\lambda_1 \neq \lambda_2 = \lambda_3$. Assume further that

$$\underbrace{\sum_{\ell=1}^{3} \mu'_\ell \, \mathbf{n}'_\ell \circ \mathbf{n}'_\ell}_{\boldsymbol{\Phi}(\boldsymbol{E}')} = \beta_o \underbrace{\sum_{\ell=1}^{3} \mathbf{n}'_\ell \circ \mathbf{n}'_\ell}_{\boldsymbol{1}} + \beta_1 \underbrace{\sum_{\ell=1}^{3} \lambda'_\ell \, \mathbf{n}'_\ell \circ \mathbf{n}'_\ell}_{\boldsymbol{E}'} \,. \tag{A.2.46}$$

This tensor equation is equivalent to two independent linear equations for the two unknowns β_o and β_1 since the third one is the same as the second:

$$\mu_1 = \beta_o + \beta_1 \lambda'_1 \,,$$
$$\mu_2 = \beta_o + \beta_1 \lambda'_2 \,.$$

The system determinant is again not zero:

$$d = \lambda'_2 - \lambda'_1 \,.$$

This means that we have unique solutions for β_o and β_1. In addition setting β_2 to zero in (A.2.45) yields

$$\boldsymbol{\Phi}(\boldsymbol{E}') = \beta_o \boldsymbol{1} + \beta_1 \boldsymbol{E}', \tag{A.2.47}$$

which shows on the basis of a comparison to (A.2.46) that representation (A.2.45) remains valid for this case as well.

(c) If $\lambda_1 = \lambda_2 = \lambda_3$ we assume that

$$\underbrace{\sum_{\ell=1}^{3} \mu'_\ell \, \mathbf{n}'_\ell \circ \mathbf{n}'_\ell}_{\boldsymbol{\Phi}(\boldsymbol{E}')} = \beta_o \underbrace{\sum_{\ell=1}^{3} \mathbf{n}'_\ell \circ \mathbf{n}'_\ell}_{\boldsymbol{1}} \,. \tag{A.2.48}$$

Hence

$$\mu'_\ell = \beta_o$$

and representation (A.2.45) simplifies to

$$\boldsymbol{\Phi}(\boldsymbol{E}') = \beta_o \boldsymbol{1} \,. \tag{A.2.49}$$

Our finial conclusion is that an isotropic tensor valued tensor function can be represented in the form

$$\boldsymbol{\Phi}(\boldsymbol{E}') = \beta_o \boldsymbol{1} + \beta_1 \boldsymbol{E}' + \beta_2 (\boldsymbol{E}')^2 \,. \tag{A.2.50}$$

where the coefficients β_0, β_1 and β_2 are functions of the invariants E'_I, E'_{II} and E'_{III}. This statement is known as the first representation theorem for isotropic tensor-tensor functions [65].

APPENDIX B

Solutions to selected problems

B.1. Problems in Chapter 1

Problem 1.1. We know the coordinates of the points A, B and C in the coordinate system $(x_1x_2x_3)$ xyz: $A(2;0;5)$ m, $B(-1;4;0)$ m, $C(-3;0;4)$ m.

(a) Determine the angle α at vertex A in the triangle ABC.

(b) Calculate the area of the triangle ABC and the volume of the tetrahedron $OABC$.

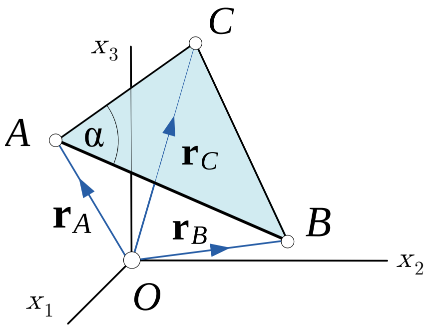

FIGURE B.1. The triangle ABC

Solution:

(a) Making use of the position vectors

$$\mathbf{r}_A = 2\mathbf{i}_1 + 5\mathbf{i}_3\,[\mathrm{m}], \quad \mathbf{r}_B = -\mathbf{i}_1 + 4\mathbf{i}_2\,[\mathrm{m}], \quad \mathbf{r}_C = -3\mathbf{i}_1 + 4\mathbf{i}_3\,[\mathrm{m}]$$

we get

$$\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A = -3\mathbf{i}_1 + 4\mathbf{i}_2 - 5\mathbf{i}_3\,[\mathrm{m}], \quad \mathbf{r}_{AC} = \mathbf{r}_C - \mathbf{r}_A = -5\mathbf{i}_1 - \mathbf{i}_3\,[\mathrm{m}]\,,$$

$$|\mathbf{r}_{AB}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.0711\,[\mathrm{m}],$$

$$|\mathbf{r}_{AC}| = \sqrt{5^2 + 1^2} = \sqrt{26} \approx 5.0990\,[\mathrm{m}].$$

It follows from the definition of the dot product (1.4) that

$$\mathbf{r}_{AB} \cdot \mathbf{r}_{AC} = |\mathbf{r}_{AB}|\,|\mathbf{r}_{AC}| \cos\alpha\,.$$

Hence

$$\cos\alpha = \frac{\mathbf{r}_{AB} \cdot \mathbf{r}_{AC}}{|\mathbf{r}_{AB}|\,|\mathbf{r}_{AC}|} = \frac{15 + 5}{7.0710 \times 5.0990} = 0.55470\,,$$

from where $\alpha = 0.982\,79$ rad $= 56.30°$.

(b) With the cross product

$$\mathbf{r}_{AB} \times \mathbf{r}_{AC} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ -3 & 4 & -5 \\ -5 & 0 & -1 \end{vmatrix} = 4\mathbf{i}_1 + 22\mathbf{i}_2 + 20\mathbf{i}_3 \ [\mathrm{m}^2]$$

the area in question is given by

$$S_{ABC} = \frac{1}{2} |\mathbf{r}_{AB} \times \mathbf{r}_{AC}| = \frac{1}{2}\sqrt{4^2 + 22^2 + 20^2} = 15 \ \mathrm{m}^2.$$

As regards the volume of the tetrahedron $OABC$ it is well known that V_{OABC} is one sixth of the volume of the parallelepiped determined by the position vectors $\mathbf{r}_A$, $\mathbf{r}_B$ and $\mathbf{r}_C$. Thus

$$V_{OABC} = \frac{1}{6}[\mathbf{r}_A \mathbf{r}_B \mathbf{r}_C] = \frac{1}{6}\begin{vmatrix} 2 & 0 & 5 \\ -1 & 4 & 0 \\ -3 & 0 & 4 \end{vmatrix} = \frac{92}{6} = 15.333 \ \mathrm{m}^3.$$

Problem 1.2. Assume that the sum of three vectors vanishes: $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Prove that

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}\,.$$

Solution: It follows from the condition the three vectors should satisfy that

$$\mathbf{b} \times \mathbf{c} = \mathbf{b} \times (-\mathbf{a} - \mathbf{b}) = -\mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = \mathbf{a} \times \mathbf{b}$$

and

$$\mathbf{c} \times \mathbf{a} = \mathbf{c} \times (-\mathbf{b} - \mathbf{c}) = -\mathbf{c} \times \mathbf{b} - \mathbf{c} \times \mathbf{c} = -\mathbf{c} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}\,.$$

Problem 1.3. Prove equations (1.16).
Solution: It is sufficient to show that the x coordinates are the same on the left and right sides of equation $(1.16)_1$ since the steps leading to the desired results are the same for the y and z coordinates.

As regards the left side we can write

$$[\mathbf{t} \times (\mathbf{u} \times \mathbf{v})] \cdot \mathbf{i}_x = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ t_x & t_y & t_z \\ \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} & \begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix} & \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \end{vmatrix} \cdot \mathbf{i}_x =$$
$$= t_y\,(u_x v_y - u_y v_x) - t_z\,(u_z v_x - u_x v_z)\ .$$

For the right side we get

$$[(\mathbf{t} \cdot \mathbf{v})\,\mathbf{u} - (\mathbf{t} \cdot \mathbf{u})\,\mathbf{v}] \cdot \mathbf{i}_x =$$
$$= \left(\underline{t_x v_x} + t_y v_y + t_z v_z\right) u_x - \left(\underline{t_x u_x} + t_y u_y + t_z u_z\right) t_x\ ,$$

which is the same as the previous result since the terms underlined cancel out. Likewise, we can prove $(1.16)_2$.

Problem 1.4. Show that the matrix

$$\underset{(3\times3)}{\mathbf{Q}} = \begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/2 \\ 1/2 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

is a proper orthogonal matrix.

Solution: If $\underset{\sim}{\mathbf{Q}}$ is proper orthogonal then $\det\left(\underset{\sim}{\mathbf{Q}}\right) = 1$ and $\underset{\sim}{\mathbf{Q}}^T \underset{\sim}{\mathbf{Q}} = \underset{\sim}{\mathbf{1}}$. It can be checked with ease that these relations are satisfied:

$$\det\left(\underset{\sim}{\mathbf{Q}}\right) = \begin{vmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = 1\,,$$

$$\underset{\sim}{\mathbf{Q}}^T \underset{\sim}{\mathbf{Q}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underset{\sim}{\mathbf{1}}\,.$$

Problem 1.5. Show that the transformation matrices are popper orthogonal matrices.
Solution: It follows from (1.27) that

$$\det(\underset{\sim}{\mathbf{Q}}) = \begin{vmatrix} \mathbf{i}'_1 \cdot \mathbf{i}_1 & \mathbf{i}'_1 \cdot \mathbf{i}_2 & \mathbf{i}'_1 \cdot \mathbf{i}_3 \\ \mathbf{i}'_2 \cdot \mathbf{i}_1 & \mathbf{i}'_2 \cdot \mathbf{i}_2 & \mathbf{i}'_2 \cdot \mathbf{i}_3 \\ \mathbf{i}'_3 \cdot \mathbf{i}_1 & \mathbf{i}'_3 \cdot \mathbf{i}_2 & \mathbf{i}'_3 \cdot \mathbf{i}_3 \end{vmatrix} = [\mathbf{i}'_1\, \mathbf{i}'_2 \mathbf{i}'_3] = 1 \tag{B.1.1}$$

since the rows of the determinant are the components of the vectors $\mathbf{i}'_\ell$ in the unprimed coordinate system. That was to be proved.
Problem 1.6. Give the unabridged form of each equation listed below. If the indicial notation is used incorrectly explain why.

$(a)\;\; F_i = G_i + H_{ij}\, a_j\,,$ $\qquad (b)\;\; u_i = v_j\,,$

$(c)\;\; F_\ell = A_\ell + B_{\ell j}\, C_j D_j\,,$ $\qquad (d)\;\; \Psi_\ell = \dfrac{\partial \Phi}{\partial x_\ell}\,,$

$(e)\;\; d = \sqrt{x_k\, x_k}\,,$ $\qquad (f)\;\; t_\alpha = \sigma_{\alpha\beta} n_\beta\,.$

Solution:

(a)

$$\begin{aligned} F_1 &= G_1 + H_{11}\, a_1 + H_{12}\, a_2 + H_{13}\, a_3\,, \\ F_2 &= G_2 + H_{21}\, a_1 + H_{22}\, a_2 + H_{23}\, a_3\,, \\ F_3 &= G_3 + H_{31}\, a_1 + H_{32}\, a_2 + H_{33}\, a_3\,. \end{aligned}$$

(b) This equation is mistaken: the free indices on the left and right sides are different: $i \neq j$.
(c) This equation is mistaken: the index j appears in the right side three times.
(d)

$$\Psi_1 = \frac{\partial \Phi}{\partial x_1}\,, \quad \Psi_2 = \frac{\partial \Phi}{\partial x_2}\,, \quad \Psi_3 = \frac{\partial \Phi}{\partial x_3}\,.$$

(e)

$$d = \sqrt{x_1\, x_1 + x_2\, x_2 + x_3\, x_3}\,.$$

(f) Keep in mind that the Greek indices take the values $1, 2$ only.

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2\,, \qquad t_2 = \sigma_{21} n_1 + \sigma_{22} n_2\,.$$

Problem 1.7. Simplify the following expressions:

$$(a)\;\; D_{pqr}\delta_{rs}\,, \quad (b)\;\; F_{k\ell m}\delta_{\ell m}\,, \quad (c)\;\; c_{pqrs}\delta_{pm}\delta_{qn}\,, \quad (d)\;\; a_{k\ell}\delta_{\ell r}\delta_{qs}\,.$$

Solution: If we take into account that the Kronecker delta is an index renaming operator we get:

$$(a)\ D_{pqr}\delta_{rs} = D_{pqs}\,, \qquad (b)\ F_{k\ell m}\delta_{\ell m} = F_{kmm}\,,$$
$$(c)\ c_{pqrs}\delta_{pm}\delta_{qn} = c_{mnrs}\,, \quad (d)\ a_{k\ell}\delta_{\ell r}\delta_{qs} = a_{kr}\delta_{qs}\,.$$

Problem 1.8. Show that equation (1.43) is equivalent to equations $(1.14)_2$.
Solution: If $r = 1$ we can write

$$e_{k\ell 1}\mathbf{i}_1 = \mathbf{i}_k \times \mathbf{i}_\ell\,.$$

The left side is different from zero if $k = 2$ and $\ell = 3$ (then $e_{231} = 1$) or $k = 3$ and $\ell = 1$ (then $e_{321} = -1$). Consequently, it holds that

$$e_{231}\mathbf{i}_1 = \mathbf{i}_1 = \mathbf{i}_2 \times \mathbf{i}_3\,, \quad \text{and} \quad e_{321}\mathbf{i}_1 = -\mathbf{i}_1 = \mathbf{i}_3 \times \mathbf{i}_2\,. \tag{B.1.2}$$

Equation $(B.1.2)_1$ is the same as the first equation in $(1.14)_2$. For $r = 2$ and $r = 3$ the proof is similar.

Problem 1.9. Show that the expression $|a_{kl}| = e_{pqr}a_{p1}a_{q2}a_{r3}$ is the expansion of the determinant by columns.
Solution: The steps leading to the solution are the same as those in equation (1.46):

$$\begin{aligned}|a_{k\ell}| &= e_{pqr}a_{p1}a_{q2}a_{r3} = \\ &= a_{11}\left(e_{1qr}a_{q2}a_{r3}\right) + a_{21}\left(e_{2qr}a_{q2}a_{r3}\right) + a_{31}\left(e_{3qr}a_{q2}a_{r3}\right) = \\ &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{21}\left(a_{13}a_{32} - a_{12}a_{33}\right) + a_{31}\left(a_{12}a_{23} - a_{13}a_{22}\right)\,.\end{aligned}$$

If we perform the first summation in q or r we get the expansion of the determinant by the second or third column.

Problem 1.10. Let $\mathbf{n}$; $|\mathbf{n}| = 1$ be the normal to the plane S that passes through the origin. Show that the component of the position vector $\mathbf{r}$ lying in the plane S is given by the mapping $\mathbf{r}_\perp = \boldsymbol{W}\cdot\mathbf{r}$ in which $\boldsymbol{W}$ is given by equation (1.207).

Solution: Figure B.2 shows the geometry of the problem.

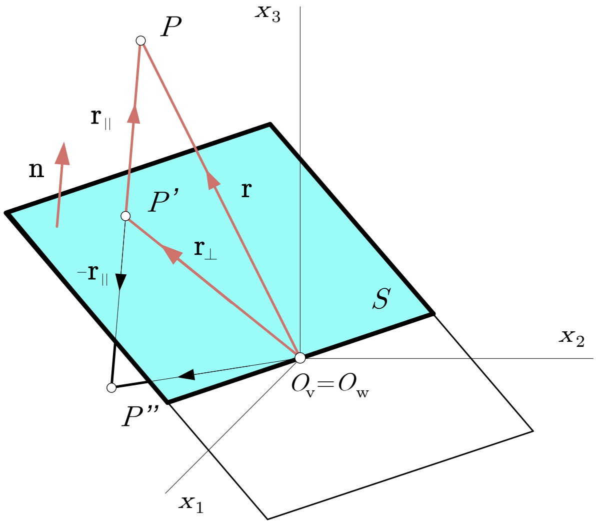

FIGURE B.2. Projection of $\mathbf{r}$ onto the plane S

It is obvious that the projection of $\mathbf{r}$ on plane S is

$$\mathbf{r}_\perp = \mathbf{r} - \mathbf{r}_\parallel = \mathbf{r} - \underbrace{\mathbf{n}\,(\mathbf{n}\cdot\mathbf{r})}_{\mathbf{r}_\parallel} .$$

Recalling the definition of the tensor product and the mapping properties of the unit tensor we can write

$$\mathbf{r}_\perp = (\mathbf{1} - \mathbf{n}\circ\mathbf{n})\cdot\mathbf{r} ,$$

from where

$$\boldsymbol{W} = \mathbf{1} - \mathbf{n}\circ\mathbf{n}$$

is the tensor sought.

Problem 1.11. Let P be the tip of the position vector $\mathbf{r}$ in the previous problem. Show that the reflection of the point P with respect to the plane S is given by the mapping $\mathbf{r}_{O_w P''} = \boldsymbol{W}\cdot\mathbf{r}$ in which $\boldsymbol{W}$ is given by equation (1.208).

Solution: It follows from Figure B.2 that

$$\mathbf{r}_{O_w P''} = \mathbf{r} - 2\mathbf{r}_\parallel = \mathbf{r} - 2\mathbf{n}\,(\mathbf{n}\cdot\mathbf{r}) = (\mathbf{1} - 2\mathbf{n}\circ\mathbf{n})\cdot\mathbf{r}$$

is the image vector. Hence

$$\boldsymbol{W} = \mathbf{1} - 2\mathbf{n}\circ\mathbf{n}$$

is the tensor of mapping.

Problem 1.12. What is the matrix of tensor (1.207)?

Solution: Since $w_{k\ell} = \mathbf{i}_k\cdot\boldsymbol{W}\cdot\mathbf{i}_\ell$ and $n_\ell = \mathbf{n}\cdot\mathbf{i}_\ell$ it easy to check that

$$\underline{\mathbf{W}} = [w_{k\ell}] = \begin{bmatrix} 1 - n_1\,n_1 & -n_1\,n_2 & -n_1\,n_3 \\ -n_2\,n_1 & 1 - n_2\,n_2 & -n_2\,n_3 \\ -n_3\,n_1 & -n_3\,n_2 & 1 - n_3\,n_3 \end{bmatrix} . \tag{B.1.3}$$

Problem 1.13. What is the matrix of tensor (1.208)?

Solution: Following the steps leading to (B.1.3) we get

$$\underline{\mathbf{W}} = [w_{k\ell}] = \begin{bmatrix} 1 - 2n_1n_1 & -2n_1n_2 & -2n_1n_3 \\ -2n_2n_1 & 1 - 2n_2n_2 & -2n_2n_3 \\ -2n_3n_1 & -2n_3n_2 & 1 - 2n_3n_3 \end{bmatrix} . \tag{B.1.4}$$

Problem 1.14. Prove equation (1.209).

Solution: Assume that we know the tensor in question in the primed coordinate system. Then we can write

$$\begin{aligned} w_{mn} = \mathbf{i}_m\cdot\boldsymbol{W}\cdot\mathbf{i}_n = \mathbf{i}_m\cdot\left(w'_{k\ell}\,\mathbf{i}'_k\circ\mathbf{i}'_\ell\right)\cdot\mathbf{i}_n = \left(\mathbf{i}_m\cdot\mathbf{i}'_k\right)w'_{k\ell}\left(\mathbf{i}'_\ell\cdot\mathbf{i}_n\right) = \\ = \underbrace{\left(\mathbf{i}_m\cdot\mathbf{i}'_k\right)}_{Q_{mk'}}\underbrace{\left(\mathbf{i}_n\cdot\mathbf{i}'_\ell\right)}_{Q_{n\ell'}}w'_{k\ell} = Q_{mk'}\,Q_{n\ell'}\,w'_{k\ell} . \end{aligned}$$

This was to be proved.

Problem 1.15. Show that relations (1.83) follow from the definition given for the transpose of a tensor.

Solution: We shall prove the last two equations only. Using the indicial notation we can write

$$w_{k\ell}\,u_\ell = u_\ell\,w_{k\ell} = u_\ell\,(w^T)_{\ell k} , \qquad u_p\,w_{pq} = w_{pq}\,u_p = (w^T)_{qp}\,u_p;$$

$$(s_{pq}\,w_{qr})^T = s_{rq}\;w_{qp} = (w^T)_{pq}\,(s^T)_{qr} .$$

Problem 1.16. Show that (a) $s_{k\ell} = s_{\ell k}$ if $\boldsymbol{S}$ is symmetric; (b) $s_{k\ell} = -s_{\ell k}$, i.e., $s_{11} = s_{22} = s_{33} = 0$, $s_{12} = -s_{21}$, $s_{13} = -s_{31}$ and $s_{23} = -s_{32}$ if $\boldsymbol{S}$ is skew.

Solution: If $\boldsymbol{S}$ is symmetric then $\boldsymbol{S} = \boldsymbol{S}^T$. Making use of equations (1.81a) and (1.84) – $\mathbf{i}_\ell$ and $\mathbf{i}_k$ corresponds to $\mathbf{u}$ and $\mathbf{v}$ – we can write

$$\mathbf{i}_k \cdot \boldsymbol{S} \cdot \mathbf{i}_\ell = \mathbf{i}_\ell \cdot \boldsymbol{S}^T \cdot \mathbf{i}_k = \mathbf{i}_\ell \cdot \boldsymbol{S} \cdot \mathbf{i}_k$$

from where we get

$$\mathbf{i}_k \cdot (s_{pq}\, \mathbf{i}_p \circ \mathbf{i}_q) \cdot \mathbf{i}_\ell = \mathbf{i}_\ell \cdot (s_{pq}\, \mathbf{i}_p \circ \mathbf{i}_q) \cdot \mathbf{i}_k$$

or

$$\delta_{kp}\, s_{pq}\, \delta_{q\ell} = \delta_{\ell p}\, s_{pq}\, \delta_{qk}$$

which means that

$$s_{k\ell} = s_{\ell k}\,.$$

If the tensor is skew then a similar line of thought yields

$$s_{k\ell} = -s_{\ell k}\,.$$

Consequently $s_{11} = s_{22} = s_{33} = 0$ and $s_{12} = -s_{21}$, $s_{23} = -s_{32}$, $s_{31} = -s_{13}$.

Problem 1.17. Assume that we know the axial vector that the belongs to the skew tensor $\boldsymbol{S} = \boldsymbol{S}_{\text{skew}}$. Show that the matrix of the tensor in terms of the components of the axial vector is given by equation (1.210).

Solution: Utilizing the properties of the permutation symbol we can check that (1.210) follows from equation (1.93):

$$\begin{aligned}
s_{11} = -e_{11r}\, s_r^{(a)} = 0\,, \quad & s_{22} = -e_{22r}\, s_r^{(a)} = 0\,, \quad s_{11} = -e_{22r}\, s_r^{(a)} = 0\,,\\
s_{12} = -e_{12r}\, s_r^{(a)} = -e_{123}\, s_3^{(a)} = -s_3^{(a)}, \quad & s_{21} = -e_{21r}\, s_r^{(a)} = -e_{213}\, s_3^{(a)} = s_3^{(a)}\,,\\
s_{23} = -e_{23r}\, s_r^{(a)} = -e_{231}\, s_1^{(a)} = -s_1^{(a)}, \quad & s_{32} = -e_{32r}\, s_r^{(a)} = -e_{321}\, s_3^{(a)} = s_1^{(a)}\,,\\
s_{31} = -e_{31r}\, s_r^{(a)} = -e_{312}\, s_2^{(a)} = -s_2^{(a)}, \quad & s_{13} = -e_{13r}\, s_r^{(a)} = -e_{132}\, s_2^{(a)} = s_2^{(a)}\,.
\end{aligned}$$

Problem 1.18. The matrix of a tensor $\boldsymbol{T}$ in the coordinate system $(x_1x_2x_3)$ is given by equation (1.211). Find the tensor components $t'_{k\ell}$ if we know the orthonormal triplet of the base vectors that belong to the coordinate system $(x'_1x'_2x'_3)$ – see equation (1.212).

Solution: Utilizing the image vectors

$$\left[\boldsymbol{T}\cdot\mathbf{i}_1^{'}\right] = \begin{bmatrix} 80 & 0 & 0 \\ 0 & 40 & -32 \\ 0 & -32 & -80 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 80 \\ 0 \\ 0 \end{bmatrix} \ \text{N/mm}^2\,,$$

$$\left[\boldsymbol{T}\cdot\mathbf{i}_2^{'}\right] = \frac{1}{\sqrt{17}} \begin{bmatrix} 80 & 0 & 0 \\ 0 & 40 & -32 \\ 0 & -32 & -80 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{17}} \begin{bmatrix} 0 \\ 192 \\ -48 \end{bmatrix} \ \text{N/mm}^2\,,$$

$$\left[\boldsymbol{T}\cdot\mathbf{i}_3^{'}\right] = \frac{1}{\sqrt{17}} \begin{bmatrix} 80 & 0 & 0 \\ 0 & 40 & -32 \\ 0 & -32 & -80 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{17}} \begin{bmatrix} 0 \\ -88 \\ -352 \end{bmatrix} \ \text{N/mm}^2$$

we have

$$t_{11}^{'} = \mathbf{i}_1^{'} \cdot \boldsymbol{T} \cdot \mathbf{i}_1^{'} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 0 \\ 0 \end{bmatrix} = 80 \ \text{N/mm}^2\,,$$

$$t'_{21} = \mathbf{i}'_2 \cdot \boldsymbol{T} \cdot \mathbf{i}'_1 = \frac{1}{\sqrt{17}} \begin{bmatrix} 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 80 \\ 0 \\ 0 \end{bmatrix} = 0 \text{ N/mm}^2 ,$$

$$t'_{31} = \mathbf{i}'_2 \cdot \boldsymbol{T} \cdot \mathbf{i}'_1 = \frac{1}{\sqrt{17}} \begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 80 \\ 0 \\ 0 \end{bmatrix} = 0 \text{ N/mm}^2 ,$$

$$t'_{12} = \mathbf{i}'_1 \cdot \boldsymbol{T} \cdot \mathbf{i}'_2 = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 192 \\ -48 \end{bmatrix} = 0 \ \text{ N/mm}^2 ,$$

$$t'_{22} = \mathbf{i}'_2 \cdot \boldsymbol{T} \cdot \mathbf{i}'_2 = \frac{1}{17} \begin{bmatrix} 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 192 \\ -48 \end{bmatrix} = 48 \text{ N/mm}^2 ,$$

$$t'_{32} = \mathbf{i}'_2 \cdot \boldsymbol{T} \cdot \mathbf{i}'_2 = \frac{1}{\sqrt{17}} \begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 192 \\ -48 \end{bmatrix} = 0 \text{ N/mm}^2 ,$$

$$t'_{13} = \mathbf{i}'_1 \cdot \boldsymbol{T} \cdot \mathbf{i}'_3 = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -88 \\ -352 \end{bmatrix} = 0 \ \text{ N/mm}^2 ,$$

$$t'_{23} = \mathbf{i}'_2 \cdot \boldsymbol{T} \cdot \mathbf{i}'_3 = \frac{1}{17} \begin{bmatrix} 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -88 \\ -352 \end{bmatrix} = 0 \text{ N/mm}^2 ,$$

$$t'_{33} = \mathbf{i}'_2 \cdot \boldsymbol{T} \cdot \mathbf{i}'_1 = \frac{1}{17} \begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -88 \\ -352 \end{bmatrix} = -88 \text{ N/mm}^2 .$$

Consequently,

$$\left[t'_{k\ell} \right] = \begin{bmatrix} 80 & 0 & 0 \\ 0 & 48 & 0 \\ 0 & 0 & -88 \end{bmatrix} \text{ N/mm}^2 .$$

Problem 1.19. Prove that

$$\boldsymbol{S} \cdot\cdot \left(\boldsymbol{T} \cdot \boldsymbol{W} \right) = \left(\boldsymbol{T}^T \cdot \boldsymbol{S} \right) \cdot\cdot \boldsymbol{W} = \left(\boldsymbol{S} \cdot \boldsymbol{W}^T \right) \cdot\cdot \boldsymbol{T} .$$

Solution: Using indicial notation we have

$$\begin{aligned} \boldsymbol{S} \cdot\cdot \left(\boldsymbol{T} \cdot \boldsymbol{W} \right) &= s_{pq} \, \left(t_{pk} \, w_{kq} \right) , \\ \left(\boldsymbol{T}^T \cdot \boldsymbol{S} \right) \cdot\cdot \boldsymbol{W} &= \left((t^T)_{kp} \, s_{pq} \right) \, w_{kq} = t_{pk} \, s_{pq} \, w_{kq} = s_{pq} \, \left(t_{pk} \, w_{kq} \right) , \\ \left(\boldsymbol{S} \cdot \boldsymbol{W}^T \right) \cdot\cdot \boldsymbol{T} &= \left(s_{pq} \, (w^T)_{qk} \right) \, t_{pk} = s_{pq} \, w_{kq} \, t_{pk} = s_{pq} \, \left(t_{pk} \, w_{kq} \right) . \end{aligned}$$

Since the right sides are equal so are the left sides.

Problem 1.20. Show that the eigenvalues and eigenvectors of the tensor

$$\boldsymbol{W} = \kappa\,(\mathbf{1} - \mathbf{i}_1 \circ \mathbf{i}_1) + \gamma\,(\mathbf{i}_1 \circ \mathbf{i}_2 + \mathbf{i}_2 \circ \mathbf{i}_1)\ .$$

are given by the following relations:

$$\lambda_1 = \frac{\kappa}{2} - \frac{1}{2}\sqrt{\kappa^2 + 4\gamma^2}\,, \qquad \mathbf{n}_1 = \frac{1}{\sqrt{1 + \left(\frac{\lambda_3}{\gamma}\right)^2}}\left(-\frac{\lambda_3}{\gamma}\mathbf{i}_1 + \mathbf{i}_2\right);$$

$$\lambda_2 = \kappa\,, \qquad \mathbf{n}_2 = \mathbf{i}_3\,;$$

$$\lambda_3 = \frac{\kappa}{2} + \frac{1}{2}\sqrt{\kappa^2 + 4\gamma^2}\,, \qquad \mathbf{n}_3 = \frac{1}{\sqrt{1 + \left(\frac{\lambda_1}{\gamma}\right)^2}}\left(-\frac{\lambda_1}{\gamma}\mathbf{i}_1 + \mathbf{i}_2\right).$$

Solution: It is obvious that the second eigenvalue and eigenvector are

$$\lambda_2 = \kappa\,, \qquad \mathbf{n}_2 = \mathbf{i}_3\,.$$

Since

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \mathbf{n}_2 \cdot \mathbf{n}_3 = 0$$

and

$$\left(-\frac{\lambda_3}{\gamma}\mathbf{i}_1 + \mathbf{i}_2\right) \cdot \left(-\frac{\lambda_1}{\gamma}\mathbf{i}_1 + \mathbf{i}_2\right) = 1 + \frac{1}{\gamma^2}\lambda_1\lambda_2 =$$
$$= 1 + \frac{1}{\gamma^2}\left(\frac{\kappa}{2} - \frac{1}{2}\sqrt{\kappa^2 + 4\gamma^2}\right)\left(\frac{\kappa}{2} + \frac{1}{2}\sqrt{\kappa^2 + 4\gamma^2}\right) = 0$$

it follows that $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are mutually perpendicular to each other. Hence it is sufficient to show that either $\mathbf{n}_1$ or $\mathbf{n}_3$ is an eigenvector. On the basis of

$$\boldsymbol{W} \cdot \mathbf{n}_1 = \frac{1}{\sqrt{1 + \left(\frac{\lambda_1}{\gamma}\right)^2}}\left[\gamma\mathbf{i}_2 \circ \mathbf{i}_1 + (\gamma\mathbf{i}_1 + \kappa\mathbf{i}_2) \circ \mathbf{i}_2 + \kappa\mathbf{i}_3 \circ \mathbf{i}_3\right] \cdot \left(\frac{\lambda_3}{\gamma}\mathbf{i}_1 + \mathbf{i}_2\right) =$$
$$= \frac{1}{\lambda_1}\frac{1}{\sqrt{1 + \left(\frac{\lambda_1}{\gamma}\right)^2}}\left(\gamma\mathbf{i}_1 + (\kappa - \lambda_3)\,\mathbf{i}_2\right) = \frac{1}{\sqrt{1 + \left(\frac{\lambda_1}{\gamma}\right)^2}}\left(\frac{\gamma}{\lambda_1}\mathbf{i}_1 + \mathbf{i}_2\right)$$

we shall calculate the cross product

$$\left(\frac{\gamma}{\lambda_1}\mathbf{i}_1 + \mathbf{i}_2\right) \times \left(-\frac{\lambda_3}{\gamma}\mathbf{i}_1 + \mathbf{i}_2\right) = \left(\frac{\gamma}{\lambda_1} + \frac{\lambda_3}{\gamma}\right)\mathbf{i}_3 =$$
$$= \frac{\gamma^2 + \lambda_1\lambda_3}{\lambda_1}\mathbf{i}_3 = \frac{1}{\lambda_1}\left[\gamma^2 + \left(\frac{\kappa}{2} - \frac{1}{2}\sqrt{\kappa^2 + 4\gamma^2}\right)\left(\frac{\kappa}{2} + \frac{1}{2}\sqrt{\kappa^2 + 4\gamma^2}\right)\right]\mathbf{i}_3 = \mathbf{0}\,.$$

This means that $\boldsymbol{W} \cdot \mathbf{n}_1$ is parallel to $\mathbf{n}_1$, i.e., $\mathbf{n}_1$ is an eigenvector.

Problem 1.21. Prove that the permutation symbol is a tensor of order three in Cartesian coordinate systems. (Hint: $e'_{k\ell r} = [\mathbf{i}'_k\,\mathbf{i}'_\ell\,\mathbf{i}'_r]$.)

Solution: Using equation (1.26c) we can write

$$e'_{ijk} = \left[\mathbf{i}'_i\,\mathbf{i}'_j\,\mathbf{i}'_k\right] = \left[Q_{i'p}\mathbf{i}_p\,Q_{j'q}\mathbf{i}_q\,Q_{k'r}\mathbf{i}_r\right] = Q_{i'p}\,Q_{j'q}\,Q_{k'r}\,[\mathbf{i}_p\,\mathbf{i}_q\,\mathbf{i}_r] = Q_{i'p}\,Q_{j'q}\,Q_{k'r}\,e_{pqr}\,. \tag{B.1.5}$$

Consequently, the permutation symbol is a tensor of order three in Cartesian coordinate systems.

Problem 1.22. Prove relationship (1.216):

Solution: Using indicial notation yields

$$\det\left(w_{k\ell} + \mathrm{d}w_{k\ell}\right) = \det\left(w_{k\ell}\right)\det\left(\delta_{p\ell} + w_{ps}^{-1}\mathrm{d}w_{s\ell}\right),$$

where we have formally an eigenvalue problem

$$\left(w_{ps}^{-1}\mathrm{d}w_{s\ell} - \lambda\delta_{p\ell}\right) n_v = 0$$

in which $\lambda = -1$.Its determinant is the characteristic equation:

$$\det\left(\delta_{p\ell} + w_{ps}^{-1}\mathrm{d}w_{s\ell}\right) = \delta_{p\ell} + \left(w_{ps}^{-1}\mathrm{d}w_{s\ell}\right)_I + \left(w_{ps}^{-1}\mathrm{d}w_{s\ell}\right)_{II} + \left(w_{ps}^{-1}\mathrm{d}w_{s\ell}\right)_{III}.$$

Terms with order higher than 1 can be dropped. Hence we get

$$\det\left(w_{k\ell} + \mathrm{d}w_{k\ell}\right) \approx \det\left(w_{k\ell}\right)\delta_{p\ell} + \underline{\det\left(w_{k\ell}\right) w_{ps}^{-1}}\mathrm{d}w_{sp}.$$

On the other hand

$$\det\left(w_{k\ell} + dw_{k\ell}\right) \approx w_{k\ell} + \underline{\frac{\partial\det\left(w_{k\ell}\right)}{\partial w_{sp}}}\mathrm{d}w_{sp}.$$

Since the terms underlined in the previous two equations are the same we obtain

$$\frac{\partial\det\left(w_{k\ell}\right)}{\partial w_{sp}} = \det\left(w_{k\ell}\right) w_{ps}^{-1}$$

or

$$\frac{\partial\det\left(\mathbf{W}\right)}{\partial\mathbf{W}} = \det\left(\mathbf{W}\right)\mathbf{W}^{-T}.$$

That was to be proved.

B.2. Problems in Chapter 2

Problem 2.1. Let

$$x_1 = X_2 + X_3 - X_1, \qquad x_2 = X_2 + X_3, \qquad x_3 = X_2 - 2X_3$$

be the motion law in material description. Determine $\boldsymbol{F}$, $\boldsymbol{F}^{-1}$ and J.

Solution: Substitute the law of motion into equation $(2.13)_2$. After performing the derivations we get the deformation gradient:

$$[F_{\ell A}] = \left[\frac{\partial\chi_\ell}{\partial X_A}\right] \underset{\chi_\ell = x_\ell}{=\uparrow=} \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & -\dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

Using (2.20) for the Jacobian we have

$$J = \det(\boldsymbol{F}) = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \end{vmatrix} = 3.$$

The equation system

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

yields the inverse motion law:

$$\chi_1^{-1} = X_1 = x_2 - x_1, \qquad \chi_2^{-1} = X_2 = \frac{1}{3}(2x_2 + x_3) \qquad \chi_3^{-1} = X_3 = \frac{1}{3}(x_2 - x_3) \,.$$

Hence

$$[F_{B\ell}^{-1}] = \left[\frac{\partial \chi_B^{-1}}{\partial x_\ell}\right] \underset{\chi_B^{-1}=X_B}{=\uparrow=} \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

is the inverse deformation gradient.

Problem 2.2. Let $\boldsymbol{\chi} = (X_1 + a\,t\,X_2)\,\mathbf{i}_1 + (X_2 - a\,t\,X_1)\,\mathbf{i}_2 + X_3\mathbf{i}_3$ be the motion law in material description where a is a constant. Find (a) the deformation gradient, (b) the inverse motion law and (c) the inverse deformation gradient.

Solution: It is easy to check that

$$[F_{\ell A}] = \left[\frac{\partial \chi_\ell}{\partial X_A}\right] = \begin{bmatrix} \frac{\partial \chi_1}{\partial X_1} & \frac{\partial \chi_1}{\partial X_2} & \frac{\partial \chi_1}{\partial X_3} \\ \frac{\partial \chi_2}{\partial X_1} & \frac{\partial \chi_2}{\partial X_2} & \frac{\partial \chi_2}{\partial X_3} \\ \frac{\partial \chi_3}{\partial X_1} & \frac{\partial \chi_3}{\partial X_2} & \frac{\partial \chi_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & a\,t & 0 \\ -a\,t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Consider now the equation system

$$\begin{bmatrix} 1 & at & 0 \\ -at & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

from where

$$\chi_1^{-1} = X_1 = \frac{-atx_2 + x_1}{1 + a^2t^2}, \qquad \chi_2^{-1} = X_2 = \frac{atx_1 + x_2}{1 + a^2t^2} \qquad \chi_3^{-1} = X_3 = x_3$$

is the inverse motion law and

$$[F_{B\ell}^{-1}] = \left[\frac{\partial \chi_B^{-1}}{\partial x_\ell}\right] \underset{\chi_B^{-1}=X_B}{=\uparrow=} \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{a^2t^2+1} & -\frac{at}{a^2t^2+1} & 0 \\ \frac{at}{a^2t^2+1} & \frac{1}{a^2t^2+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the inverse deformation gradient.

Problem 2.3. Given the temperature distribution in terms of the material coordinates X_A within the cube of Exercise 2.1: $\theta = B\,(1 + X_1X_2)$ where B is a constant. Find the spatial description of the temperature field.

Solution: Since

$$X_1 = x_1 - a_1 \left(x_2 - a_2\right)^2 \ , \quad X_2 = x_2 - a_2 \, , \quad X_3 = \frac{x_3}{1 - a_3 \left(x_2 - a_2\right)}$$

– see the solution to Exercise 2.1 – we get the temperature distribution within the cube in spacial description in the form

$$\theta = B \left(1 + X_1 \, X_2\right) = B \left[1 + \left(x_1 - a_1 \left(x_2 - a_2\right)^2\right) \, \left(x_2 - a_2\right)\right] \, .$$

Problem 2.4. Find the displacement field in material and spatial descriptions within the cube of Exercise 2.1

Solution: Using the motion law given in Exercise 2.1 we may easily determine the displacements in material description:

$$u_1 = x_1 - X_1 = a_1 X_2^2, \quad u_2 = x_2 - X_2 = a_2, \quad u_3 = x_3 - X_3 = a_3 X_2 \, X_3 \, . \qquad \text{(B.2.6)}$$

If we now substitute the inverse law of motion (2.5) we get the displacement components in spatial description:

$$u_1 = x_1 - X_1 = a_1 \left(x_2 - a_2\right)^2 , \quad u_2 = x_2 - X_2 = a_2 ,$$

$$u_3 = x_3 - X_3 = x_3 - \frac{x_3}{1 + a_3 \left(x_2 - a_2\right)} = \frac{a_3 x_3 (x_2 - a_2)}{1 + a_3 (x_2 - a_2)} \, .$$

Problem 2.5. Making use of the motion law of Exercise 2.1 determine, within the cube, (a) the deformation gradient in material and spatial descriptions and (b) prove that equation (2.98) is the inverse deformation gradient in spatial description.

Solution: Substitute the law of motion (2.4) into equation $(2.13)_2$. After performing the derivations we obtain the deformation gradient:

$$[F_{\ell A}] = \left[\frac{\partial \chi_\ell}{\partial X_A}\right] \underset{\chi_\ell = x_\ell}{=\uparrow=} \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 2a_1 X_2 & 0 \\ 0 & 1 & 0 \\ 0 & a_3 X_3 & 1 + a_3 X_2 \end{bmatrix} .$$

With the inverse law of motion (2.5) we get from $(2.16)_2$ that

$$[F^{-1}_{B\ell}] = \left[\frac{\partial \chi^{-1}_B}{\partial x_\ell}\right] \underset{\chi^{-1}_B = X_B}{=\uparrow=} \begin{bmatrix} \dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_1}{\partial x_2} & \dfrac{\partial X_1}{\partial x_3} \\ \dfrac{\partial X_2}{\partial x_1} & \dfrac{\partial X_2}{\partial x_2} & \dfrac{\partial X_2}{\partial x_3} \\ \dfrac{\partial X_3}{\partial x_1} & \dfrac{\partial X_3}{\partial x_2} & \dfrac{\partial X_3}{\partial x_3} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -2a_1 \left(x_2 - a_2\right) & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{-a_3 x_3}{\left[1 + a_3 \left(x_2 - a_2\right)\right]^2} & \dfrac{1}{1 + a_3 \left(x_2 - a_2\right)} \end{bmatrix} .$$

Problem 2.6. Find the deformation gradient in the cylindrical coordinate system.

Solution: The cylindrical coordinates are denoted by R, Θ and Z in the initial configuration. In the current configuration, however, by r, ϑ and z. The motion law is of the form

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \boldsymbol{\chi}(R, \Theta, Z)\,.$$

It is clear that

$$r = r(R, \Theta, Z)\,, \qquad \vartheta = \vartheta(R, \Theta, Z)\,, \qquad z = z(R, \Theta, Z)\,.$$

The position vector of the material point in the (initial) [current] configuration is given by $(\mathbf{R} = R\mathbf{i}_R + Z\mathbf{i}_Z)$ $[\mathbf{r} = r\mathbf{i}_r + z\mathbf{i}_z]$, that is, $(\mathbf{X} = \mathbf{R})$ $[\boldsymbol{\chi} = \mathbf{x} = \mathbf{r}]$. It holds that

$$\begin{aligned}
\mathrm{d}\mathbf{x} &= \mathrm{d}\mathbf{r} = \\
&= \frac{\partial \mathbf{r}}{\partial r}\left(\frac{\partial r}{\partial R}dR + \frac{\partial r}{R\partial \Theta}Rd\Theta + \frac{\partial r}{\partial Z}dZ\right) + \frac{\partial \mathbf{r}}{\partial \vartheta}\left(\frac{\partial \vartheta}{\partial R}dR + \frac{\partial \vartheta}{R\partial \Theta}Rd\Theta + \frac{\partial \vartheta}{\partial Z}dZ\right) + \\
&\quad + \frac{\partial \mathbf{r}}{\partial z}\left(\frac{\partial z}{\partial R}dR + \frac{\partial z}{R\partial \Theta}Rd\Theta + \frac{\partial z}{\partial Z}dZ\right) \underset{d\mathbf{R}=dR\mathbf{i}_R+Rd\Theta\mathbf{i}_\Theta+dZ\mathbf{i}_Z}{=\uparrow=} \\
&= \underbrace{\frac{\partial \mathbf{r}}{\partial r}}_{\mathbf{i}_r}\left(\frac{\partial r}{\partial R}\mathbf{i}_R\cdot d\mathbf{R} + \frac{\partial r}{R\partial \Theta}\mathbf{i}_\Theta\cdot d\mathbf{R} + \frac{\partial r}{\partial Z}\mathbf{i}_Z\cdot d\mathbf{R}\right) + \\
&\quad + \underbrace{\frac{\partial \mathbf{r}}{\partial \vartheta}}_{r\mathbf{i}_\vartheta}\left(\frac{\partial \vartheta}{\partial R}\mathbf{i}_R\cdot d\mathbf{R} + \frac{\partial \vartheta}{R\partial \Theta}\mathbf{i}_\Theta\cdot d\mathbf{R} + \frac{\partial \vartheta}{\partial Z}\mathbf{i}_Z\cdot d\mathbf{R}\right) + \\
&\quad + \underbrace{\frac{\partial \mathbf{r}}{\partial z}}_{\mathbf{i}_z}\left(\frac{\partial z}{\partial R}\mathbf{i}_R\cdot d\mathbf{R} + \frac{\partial z}{R\partial \Theta}\mathbf{i}_\Theta\cdot d\mathbf{R} + \frac{\partial z}{\partial Z}\mathbf{i}_Z\cdot d\mathbf{R}\right).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\mathrm{d}\mathbf{r} = \Bigg[&\mathbf{i}_r \otimes \left(\frac{\partial r}{\partial R}\mathbf{i}_R + \frac{\partial r}{R\partial \Theta}\mathbf{i}_\Theta + \frac{\partial r}{\partial Z}\mathbf{i}_Z\right) + r\mathbf{i}_\vartheta \otimes \left(\frac{\partial \vartheta}{\partial R}\mathbf{i}_R + \frac{\partial \vartheta}{R\partial \Theta}\mathbf{i}_\Theta + \frac{\partial \vartheta}{\partial Z}\mathbf{i}_Z\right) + \\
&+ \mathbf{i}_z \otimes \left(\frac{\partial z}{\partial R}\mathbf{i}_R + \frac{\partial z}{R\partial \Theta}\mathbf{i}_\Theta + \frac{\partial z}{\partial Z}\mathbf{i}_Z\right)\Bigg] \cdot d\mathbf{R}\,,
\end{aligned}$$

from where, after some rearrangement, we get the deformation gradient:

$$\begin{aligned}
\mathbf{F} = &\left(\frac{\partial r}{\partial R}\mathbf{i}_r + r\frac{\partial \vartheta}{\partial R}\mathbf{i}_\vartheta + \frac{\partial z}{\partial R}\mathbf{i}_Z\right) \otimes \mathbf{i}_R + \left(\frac{\partial r}{\partial \Theta}\mathbf{i}_r + r\frac{\partial \vartheta}{\partial \Theta}\mathbf{i}_\vartheta + \frac{\partial z}{\partial \Theta}\mathbf{i}_z\right) \otimes \frac{1}{R}\mathbf{i}_\Theta + \\
&+ \left(\frac{\partial r}{\partial Z}\mathbf{i}_r + r\frac{\partial \vartheta}{\partial \Theta}\mathbf{i}_\vartheta + \frac{\partial z}{\partial Z}\mathbf{i}_z\right) \otimes \mathbf{i}_Z\,. \qquad \text{(B.2.7a)}
\end{aligned}$$

Its matrix is given by

$$\underline{\mathbf{F}} = \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r\frac{\partial \vartheta}{\partial R} & \frac{r}{R}\frac{\partial \vartheta}{\partial \Theta} & r\frac{\partial \vartheta}{\partial \Theta} \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix}. \qquad \text{(B.2.7b)}$$

Problem 2.7. Assume that $r = R$, $\vartheta = \Theta + z\hat{\vartheta}$ and $z = Z$ in the previous Problem. This is the case for a rod with circular cross section subjected to twisting – it is assumed that the cross sections rotate in their own plane. Then $\hat{\vartheta}$ is the angle of rotation for a unit length. Find the displacement gradient utilizing the notations given in Figure 4.1.

Solution: The displacement vector is

$$\mathbf{u} = \mathbf{u}^\circ = r\mathbf{i}_r - R\mathbf{i}_R\,.$$

Hence

$$\begin{aligned}
\mathbf{u}^\circ \otimes \nabla^\circ = (r\mathbf{i}_r - R\mathbf{i}_R) \otimes \left(\frac{\partial}{\partial R}\mathbf{i}_R + \frac{1}{R}\frac{\partial}{\partial \Theta}\mathbf{i}_\Theta + \frac{\partial}{\partial Z}\mathbf{i}_Z\right) = \\
= \left(r\cos\hat{\vartheta}z\,\mathbf{i}_R + r\sin\hat{\vartheta}z\,\mathbf{i}_\Theta - R\mathbf{i}_R\right) \otimes \frac{\partial}{\partial R}\mathbf{i}_R + \\
+ \left(r\cos\hat{\vartheta}z\,\mathbf{i}_R + r\sin\hat{\vartheta}z\,\mathbf{i}_\Theta - R\mathbf{i}_R\right) \otimes \frac{1}{R}\frac{\partial}{\partial \Theta}\mathbf{i}_\Theta \\
+ \left(r\cos\hat{\vartheta}z\,\mathbf{i}_R + r\sin\hat{\vartheta}z\,\mathbf{i}_\Theta - R\mathbf{i}_R\right) \otimes \frac{\partial}{\partial Z}\mathbf{i}_Z = \\
= \frac{\partial r}{\partial R}\left(\cos\hat{\vartheta}Z\,\mathbf{i}_R + \sin\hat{\vartheta}Z\,\mathbf{i}_\Theta\right) \otimes \mathbf{i}_R - \mathbf{i}_R \otimes \mathbf{i}_R + \\
+ \frac{r}{R}\left(\cos\hat{\vartheta}Z\,\mathbf{i}_\Theta - \sin\hat{\vartheta}Z\,\mathbf{i}_R\right) \otimes \mathbf{i}_\Theta - \mathbf{i}_\Theta \otimes \mathbf{i}_\Theta + \\
+ r\hat{\vartheta}\left(-\sin\hat{\vartheta}Z\,\mathbf{i}_R + \cos\hat{\vartheta}Z\,\mathbf{i}_\Theta\right) \otimes \mathbf{i}_Z\,,
\end{aligned}$$

where the relations $z = Z$ and $\mathbf{i}_z = \mathbf{i}_Z$ are taken into account. Since $r = R$ we also have that $\partial r/\partial R = r/R = 1$. Thus

$$\begin{aligned}
\mathbf{u}^\circ \otimes \nabla^\circ = \left(\cos\hat{\vartheta}Z\,\mathbf{i}_R + \sin\hat{\vartheta}Z\,\mathbf{i}_\Theta\right) \otimes \mathbf{i}_R - \mathbf{i}_R \otimes \mathbf{i}_R + \\
+ \left(\cos\hat{\vartheta}Z\,\mathbf{i}_\Theta - \sin\hat{\vartheta}Z\,\mathbf{i}_R\right) \otimes \mathbf{i}_\Theta - \mathbf{i}_\Theta \otimes \mathbf{i}_\Theta + \\
+ R\hat{\vartheta}\left(-\sin\hat{\vartheta}Z\,\mathbf{i}_R + \cos\hat{\vartheta}Z\,\mathbf{i}_\Theta\right) \otimes \mathbf{i}_Z
\end{aligned}$$

and

$$[\mathbf{u}^\circ \otimes \nabla^\circ] = \begin{bmatrix} \cos\hat{\vartheta}Z - 1 & -\sin\hat{\vartheta}Z & -R\hat{\vartheta}\sin\hat{\vartheta}Z \\ \sin\hat{\vartheta}Z & \cos\hat{\vartheta}Z - 1 & R\hat{\vartheta}\cos\hat{\vartheta}Z \\ 0 & 0 & 0 \end{bmatrix}.$$

If $\hat{\vartheta}Z$ and $\hat{\vartheta}R$ are very small (for the case of small deformations) the matrix of the dispalcement gradient is simplified to

$$[\mathbf{u}^\circ \otimes \nabla^\circ] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & R\hat{\vartheta} \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 2.8. Show that equation (2.99) provides the Green-Lagrange strain tensor for the motion given in Exercise 2.1.

Solution: Since

$$u_1 = a_1 X_2^2, \quad u_2 = a_2, \quad u_3 = a_3 X_2\, X_3$$

equation (2.39) yields

$$E_{11} = \frac{1}{2}\left(u_{1,1} + u_{1,1} + u_{1,1}u_{1,1} + u_{2,1}u_{2,1} + u_{3,1}u_{3,1}\right) = 0\,,$$
$$E_{22} = \frac{1}{2}\left(u_{2,2} + u_{2,2} + u_{1,2}u_{1,2} + u_{2,2}u_{2,2} + u_{3,2}u_{3,2}\right) = 2a_1^2 X_2^2 + \frac{1}{2}a_3^2 X_3^2\,,$$
$$E_{33} = \frac{1}{2}\left(u_{3,3} + u_{3,3} + u_{1,3}u_{1,3} + u_{2,3}u_{2,3} + u_{3,3}u_{3,3}\right) = a_3 X_2 + \frac{1}{2}a_3^2 X_2^2\,,$$
$$E_{12} = E_{21} = \frac{1}{2}\left(u_{1,2} + u_{2,1} + u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}\right) = a_1 X_2\,,$$
$$E_{23} = E_{32} = \frac{1}{2}\left(u_{2,3} + u_{3,2} + u_{1,2}u_{1,3} + u_{2,2}u_{2,3} + u_{3,2}u_{3,3}\right) = \frac{1}{2}a_3 X_3 + \frac{1}{2}a_3^2 X_2 X_3\,,$$
$$E_{31} = E_{13} = \frac{1}{2}\left(u_{3,1} + u_{1,3} + u_{1,1}u_{1,3} + u_{2,1}u_{2,3} + u_{3,1}u_{3,3}\right) = 0\,.$$

Hence

$$\underset{(3\times3)}{\underline{\mathbf{E}}} = [E_{AB}] = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & a_1 X_2 & 0 \\ a_1 X_2 & 2a_1^2 X_2^2 + \frac{1}{2}a_3^2 X_3^2 & \frac{1}{2}a_3 X_3\left(1 + a_3 X_2\right) \\ 0 & \frac{1}{2}a_3 X_3\left(1 + a_3 X_2\right) & a_3 X_2\left(1 + \frac{1}{2}a_3 X_2\right) \end{bmatrix}. \tag{B.2.8}$$

Problem 2.9. For the deformation

$$x_1 = X_1\,, \qquad x_2 = -3X_3\,, \qquad x_3 = 2X_2$$

find $\boldsymbol{F}$, $\boldsymbol{U}$, $\boldsymbol{v}$ and $\boldsymbol{R}$.

Solution: It can be checked easily that

$$\underline{\mathbf{F}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix}, \qquad \det\left(\underline{\mathbf{F}}\right) = 6 > 0\,.$$

Consequently,

$$\underline{\mathbf{C}} = \underline{\mathbf{U}}^2 = \underline{\mathbf{F}}^T \underline{\mathbf{F}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Note that $\underline{\boldsymbol{C}} = \underline{\boldsymbol{U}}^2$ is diagonal. Hence

$$\underline{\mathbf{U}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \underline{\mathbf{U}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix},$$

and

$$\underline{\mathbf{R}} = \underline{\mathbf{F}}\,\underline{\mathbf{U}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

while

$$\det(\underline{\mathbf{R}}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = 1,$$

$$\underline{\mathbf{R}}\,\underline{\mathbf{R}}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For $\boldsymbol{v}$ we get

$$\underline{\boldsymbol{v}} = \underline{\mathbf{F}}\,\underline{\mathbf{R}}^{-1} = \underline{\mathbf{F}}\,\underline{\mathbf{R}}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Problem 2.10. For the deformation

$$x_1 = 2X_1 - 2X_2\,, \qquad x_2 = X_1 + X_2, \qquad x_3 = X_3$$

find $\boldsymbol{F}$, $\boldsymbol{U}$, $\boldsymbol{v}$ and $\boldsymbol{R}$. Prove that the matrix of the left stretch tensor $\boldsymbol{v}$ is a diagonal matrix.

Solution: It can be checked easily that

$$\underline{\mathbf{F}} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \det(\underline{\mathbf{F}}) = 4 > 0\,.$$

Consequently,

$$\underline{\mathbf{C}} = \underline{\mathbf{F}}^T\underline{\mathbf{F}} = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution for the eigenvalue problem for the right Cauchy-Green strain tensor $\boldsymbol{C}$ ($\chi_i = \lambda_i^2$) is based on the observation that $\mathbf{i}_3$ is an eigenvector with $\chi = 1$ as the eigenvalue. Hence it is sufficient to consider the eigenvalue problem for the matrix

$$\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}.$$

This means that

$$P_2(\chi) = \begin{vmatrix} 5-\chi & -3 \\ -3 & 5-\chi \end{vmatrix} = \chi^2 - 10\chi + 16 = 0$$

is the characteristic equation from where 8 and 2 are the eigenvalues. With the eigenvalues

$$\chi_1 = \lambda_1^2 = 8, \quad \chi_2 = \lambda_2^2 = 2, \quad \chi_3 = \lambda_3^2 = 1;$$

$$\underset{(n^\circ)}{\underline{\boldsymbol{U}}}{}^2 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underset{(n^\circ)}{\underline{\boldsymbol{U}}} = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underset{(n^\circ)}{\underline{\boldsymbol{U}}}{}^{-1} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix}.$$

As regards the eigenvectors if $\chi_1 = 8$ we have to solve the equation system

$$\begin{bmatrix} 5-\chi_1 & -3 \\ -3 & 5-\chi_2 \end{bmatrix} \begin{bmatrix} n^\circ_{11} \\ n^\circ_{21} \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} n^\circ_{11} \\ n^\circ_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

from where we get

$$\mathbf{n}^\circ_1 = \frac{1}{\sqrt{2}}\mathbf{i}_1 - \frac{1}{\sqrt{2}}\mathbf{i}_2,$$

$$\mathbf{n}^\circ_2 = \mathbf{n}^\circ_3 \times \mathbf{n}^\circ_1 = \mathbf{i}_3 \times \left(\frac{1}{\sqrt{2}}\mathbf{i}_1 - \frac{1}{\sqrt{2}}\mathbf{i}_2\right) = \frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{1}{\sqrt{2}}\mathbf{i}_2$$

and as we have already seen

$$\mathbf{n}^\circ_3 = \mathbf{i}_3.$$

Consequently,

$$\begin{aligned} \boldsymbol{U} = \sum_{\ell=1}^{3} \lambda_\ell \mathbf{n}^\circ_\ell \circ \mathbf{n}^\circ_\ell &= 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\mathbf{i}_1 - \frac{1}{\sqrt{2}}\mathbf{i}_2\right) \circ \left(\frac{1}{\sqrt{2}}\mathbf{i}_1 - \frac{1}{\sqrt{2}}\mathbf{i}_2\right) + \\ &+ \sqrt{2}\left(\frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{1}{\sqrt{2}}\mathbf{i}_2\right) \circ \left(\frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{1}{\sqrt{2}}\mathbf{i}_2\right) + \mathbf{i}_3 \circ \mathbf{i}_3 = \\ &= \left(\frac{3}{\sqrt{2}}\mathbf{i}_1 - \frac{1}{\sqrt{2}}\mathbf{i}_2\right) \circ \mathbf{i}_1 + \left(-\frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{3}{\sqrt{2}}\mathbf{i}_2\right) \circ \mathbf{i}_2 + \mathbf{i}_3 \circ \mathbf{i}_3 \end{aligned}$$

and

$$\underline{\mathbf{U}} = \begin{bmatrix} \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

from where

$$\underline{\mathbf{U}}^2 = \begin{bmatrix} \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\mathbf{C}}.$$

As regards the inverse of $\boldsymbol{U}$ we have

$$\begin{aligned} \boldsymbol{U}^{-1} = \sum_{\ell=1}^{3} \frac{1}{\lambda_\ell} \mathbf{n}^\circ_\ell \circ \mathbf{n}^\circ_\ell &= \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}}\mathbf{i}_1 - \frac{1}{\sqrt{2}}\mathbf{i}_2\right) \circ \left(\frac{1}{\sqrt{2}}\mathbf{i}_1 - \frac{1}{\sqrt{2}}\mathbf{i}_2\right) + \\ &+ \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{1}{\sqrt{2}}\mathbf{i}_2\right) \circ \left(\frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{1}{\sqrt{2}}\mathbf{i}_2\right) + \mathbf{i}_3 \circ \mathbf{i}_3 = \\ &= \left(\frac{3}{4\sqrt{2}}\mathbf{i}_1 + \frac{1}{4\sqrt{2}}\mathbf{i}_2\right) \circ \mathbf{i}_1 + \left(\frac{1}{4\sqrt{2}}\mathbf{i}_1 + \frac{3}{4\sqrt{2}}\mathbf{i}_2\right) \circ \mathbf{i}_2 + \mathbf{i}_3 \circ \mathbf{i}_3 \end{aligned}$$

and

$$\underline{\mathbf{U}}^{-1} = \frac{1}{4\sqrt{2}} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4\sqrt{2} \end{bmatrix}.$$

The matrix of the rotation tensor is

$$\underline{\mathbf{R}} = \underline{\mathbf{F}}\,\underline{\mathbf{U}}^{-1} = \frac{1}{4\sqrt{2}} \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that

$$\det(\underline{\mathbf{R}}) = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

$$\underline{\mathbf{R}}\,\underline{\mathbf{R}}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For the matrix of the left stretch tensor $\boldsymbol{v}$ we get

$$\underline{\boldsymbol{v}} = \underline{\mathbf{F}}\,\underline{\mathbf{R}}^{-1} = \underline{\mathbf{F}}\,\underline{\mathbf{R}}^T = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It also holds that

$$\underline{\boldsymbol{v}}\,\underline{\mathbf{R}} = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\mathbf{F}}.$$

It is obvious that the matrix of the left stretch tensor $\boldsymbol{v}$ is a diagonal matrix.

Problem 2.11. For the deformation

$$x_1 = 2X_3\,, \qquad x_2 = -X_1\,, \qquad x_3 = -2X_2 + 3X_3$$

determine $\boldsymbol{F}$, $\boldsymbol{U}$, $\boldsymbol{v}$ and $\boldsymbol{R}$.

Solution: It is clear that

$$\underline{\mathbf{F}} = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & -2 & 3 \end{bmatrix}, \qquad \det(\underline{\mathbf{F}}) = 4 > 0\,.$$

Hence

$$\underline{\mathbf{C}} = \underline{\mathbf{U}}^2 = \underline{\mathbf{F}}^T\underline{\mathbf{F}} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -6 \\ 0 & -6 & 13 \end{bmatrix}.$$

We do not give details concerning the eigenvalue problem of the tensor $\boldsymbol{C}$. It can be checked that

$$\lambda_1^2 = 16\,, \qquad \lambda_2^2 = \lambda_3^2 = 1$$

are the three eigenvalues and

$$\mathbf{n}_1^o = \frac{1}{\sqrt{5}}\left(-\mathbf{i}_2 + 2\mathbf{i}_3\right), \qquad \mathbf{n}_2^o = \mathbf{i}_1\,, \qquad \mathbf{n}_3^o = \frac{1}{\sqrt{5}}\left(2\mathbf{i}_2 + \mathbf{i}_3\right)$$

are the associated eigenvectors. Since

$$\underset{(n)}{\boldsymbol{U}} = \lambda_1\mathbf{n}_1^o \circ \mathbf{n}_1^o + \lambda_2\mathbf{n}_2^o \circ \mathbf{n}_2^o + \lambda_3\mathbf{n}_3^o \circ \mathbf{n}_3^o = 4\mathbf{n}_1^o \circ \mathbf{n}_1^o + \mathbf{n}_2^o \circ \mathbf{n}_2^o + \mathbf{n}_3^o \circ \mathbf{n}_3^o =$$
$$= \frac{4}{5}\left(-\mathbf{i}_2 + 2\mathbf{i}_3\right) \circ \left(-\mathbf{i}_2 + 2\mathbf{i}_3\right) + \mathbf{i}_1 \circ \mathbf{i}_1 + \frac{1}{5}\left(2\mathbf{i}_2 + \mathbf{i}_3\right) \circ \left(2\mathbf{i}_2 + \mathbf{i}_3\right) =$$
$$= \mathbf{i}_1 \circ \mathbf{i}_1 + \frac{1}{5}\left(4\mathbf{i}_2 - 8\mathbf{i}_3 + 4\mathbf{i}_2 + 2\mathbf{i}_3\right) \circ \mathbf{i}_2 + \frac{1}{5}\left(-8\mathbf{i}_2 + 16\mathbf{i}_3 + 2\mathbf{i}_2 + \mathbf{i}_3\right) \circ \mathbf{i}_3 =$$
$$= \mathbf{i}_1 \circ \mathbf{i}_1 + \frac{1}{5}\left(8\mathbf{i}_2 - 6\mathbf{i}_3\right) \circ \mathbf{i}_2 + \frac{1}{5}\left(-6\mathbf{i}_2 + 17\mathbf{i}_3\right) \circ \mathbf{i}_3$$

we get that

$$\underline{\mathbf{U}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{8}{5} & -\frac{6}{5} \\ 0 & -\frac{6}{5} & \frac{17}{5} \end{bmatrix}, \qquad \underline{\mathbf{U}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{17}{20} & \frac{3}{10} \\ 0 & \frac{3}{10} & \frac{2}{5} \end{bmatrix}.$$

Thus

$$\underline{\mathbf{R}} = \underline{\mathbf{F}}\,\underline{\mathbf{U}}^{-1} = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{17}{20} & \frac{3}{10} \\ 0 & \frac{3}{10} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -1 & 0 & 0 \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

It can be checked with ease that

$$\det(\underline{\mathbf{R}}) = \begin{vmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -1 & 0 & 0 \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{vmatrix} = 1,$$

$$\underline{\mathbf{R}}\,\underline{\mathbf{R}}^T = \begin{bmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -1 & 0 & 0 \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ \frac{3}{5} & 0 & -\frac{4}{5} \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As regards $\boldsymbol{v}$ we have

$$\underline{\boldsymbol{v}} = \underline{\mathbf{F}}\,\underline{\mathbf{R}}^{-1} = \underline{\mathbf{F}}\,\underline{\mathbf{R}}^{T} = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ \frac{3}{5} & 0 & -\frac{4}{5} \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{8}{5} & 0 & \frac{6}{5} \\ 0 & 1 & 0 \\ \frac{6}{5} & 0 & \frac{17}{5} \end{bmatrix}.$$

Problem 2.12. Given the deformation in the following form:

$$x_1 = \sqrt{2X_1}\cos X_2\,, \qquad x_2 = \sqrt{2X_1}\sin X_2\,, \qquad x_3 = X_3\,.$$

Find the inverse motion law, the deformation gradients $\boldsymbol{F}$, $\boldsymbol{F}^{-1}$ and show that the above deformation is volume preserving.

Solution: Since

$$x_1^2 + x_2^2 = 2X_1\,, \qquad \frac{x_2}{x_1} = \tan X_2$$

for $x_1^2 + x_2^2 > 0$ the inverse motion law is given by the following equations

$$X_1 = \frac{1}{2}\left(x_1^2 + x_2^2\right)\,, \qquad X_2 = \tan^{-1}\frac{x_2}{x_1}\,, \qquad X_3 = x_3\,.$$

Hence

$$[F_{\ell A}] = \left[\frac{\partial \chi_\ell}{\partial X_A}\right] \underset{\chi_\ell = x_\ell}{=\uparrow=} \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & -\frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2X_1}}\cos X_2 & -\sqrt{2X_1}\sin X_2 & 0 \\ \frac{1}{\sqrt{2X_1}}\sin X_2 & \sqrt{2X_1}\cos X_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$[F^{-1}_{B\ell}] = \left[\frac{\partial \chi_B^{-1}}{\partial x_\ell}\right] \underset{\chi_B^{-1}=X_B}{=\!\!\uparrow\!\!=} \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & 0 \\ -\frac{x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

As regards the Jacobian we get

$$J = \det(F_{\ell A}) = \cos^2 X_2 + \sin^2 X_2 = \det(F^{-1}_{B\ell}) = 1 \, .$$

Consequently the deformation is volume preserving.

Problem 2.13. For the deformation considered in Problem **2.10** find the right Cauchy-Green tensor, the Green-Lagrange strain tensor, the principal directions and stretches in the initial configuration, the right stretch tensor, the rotation tensor and the principal directions in the current configuration.

Solution: Using (2.32) we obtain

$$[C_{AB}] = [F_{A\ell} F_{\ell B}] =$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2X_1}}\cos X_2 & \frac{1}{\sqrt{2X_1}}\sin X_2 & 0 \\ -\sqrt{2X_1}\sin X_2 & \sqrt{2X_1}\cos X_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2X_1}}\cos X_2 & -\sqrt{2X_1}\sin X_2 & 0 \\ \frac{1}{\sqrt{2X_1}}\sin X_2 & \sqrt{2X_1}\cos X_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2X_1} & 0 & 0 \\ 0 & 2X_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

$$[E_{AB}] = \frac{1}{2}\left[C_{AB} - \delta_{AB}\right] = \begin{bmatrix} \frac{1}{2X_1} - 1 & 0 & 0 \\ 0 & 2X_1 - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Since the matrix $[C_{AB}]$ is diagonal equation (2.62) yields

$$\begin{aligned} \mathbf{n}_1^\circ &= \mathbf{i}_1 \, , & \lambda_1^e &= \frac{1}{\sqrt{2X_1}} \, ; \\ \mathbf{n}_2^\circ &= \mathbf{i}_2 \, , & \lambda_2^e &= \sqrt{2X_1} \, ; \\ \mathbf{n}_3^\circ &= \mathbf{i}_3 \, , & \lambda_3^e &= 1 \, . \end{aligned}$$

From equations (2.71) and (2.72) we get the right stretch tensor, its inverse and the rotation tensor:

$$[U_{AB}] = \sqrt{[C_{AB}]} = \begin{bmatrix} \frac{1}{\sqrt{2X_1}} & 0 & 0 \\ 0 & \sqrt{2X_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} , \qquad [U^{-1}_{AB}] = \begin{bmatrix} \sqrt{2X_1} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2X_1}} & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

$$[R_{kB}]=[F_{kA}U_{AB}^{-1}]=\begin{bmatrix}\frac{1}{\sqrt{2X_1}}\cos X_2 & -\sqrt{2X_1}\sin X_2 & 0\\ \frac{1}{\sqrt{2X_1}}\sin X_2 & \sqrt{2X_1}\cos X_2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2X_1} & 0 & 0\\ 0 & \frac{1}{\sqrt{2X_1}} & 0\\ 0 & 0 & 1\end{bmatrix}=$$

$$=\begin{bmatrix}\cos X_2 & -\sin X_2 & 0\\ \sin X_2 & \cos X_2 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The prinicpal directions in the current configuration can be obtained from equation (2.79):

$$\begin{aligned}\mathbf{n}_1 &= \boldsymbol{R}\cdot\mathbf{n}_1^\circ = \mathbf{i}_1\cos X_2 - \mathbf{i}_2\sin X_2\,,\\ \mathbf{n}_2 &= \boldsymbol{R}\cdot\mathbf{n}_2^\circ = \mathbf{i}_1\sin X_2 + \mathbf{i}_2\cos X_2\,,\\ \mathbf{n}_3 &= \boldsymbol{R}\cdot\mathbf{n}_3^\circ = \mathbf{i}_3\,.\end{aligned}$$

Problem 2.14. Show that the Green Lagrange strain tensor and the Euler-Almansi strain tensor are independent of the rigid body rotation (of the tensor $\boldsymbol{R}$). (Hint: Make use of the polar decomposition theorem.)

Solution: Substitute the right polar decomposition $\boldsymbol{F}=\boldsymbol{R}\cdot\boldsymbol{U}$ into the definition of the right Cauchy-Green tensor. We have

$$\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F}=\boldsymbol{U}^T\cdot\underbrace{\boldsymbol{R}^T\cdot\boldsymbol{R}}_{\boldsymbol{I}}\cdot\boldsymbol{U}=\boldsymbol{U}^T\cdot\boldsymbol{U}=(\boldsymbol{U})^2\ .$$

which means that $\boldsymbol{C}$ is independent of $\boldsymbol{R}$. Hence

$$\boldsymbol{E}=\frac{1}{2}(\boldsymbol{C}-\boldsymbol{1})$$

is also independent of $\boldsymbol{R}$.

As regards the second part of our statement take into account the left polar decomposition $\boldsymbol{F}=\boldsymbol{v}\cdot\boldsymbol{R}$. We may write

$$\boldsymbol{b}=\boldsymbol{F}\cdot\boldsymbol{F}^T=\boldsymbol{v}\cdot\boldsymbol{R}\cdot(\boldsymbol{v}\cdot\boldsymbol{R})^T=\boldsymbol{v}\cdot\underbrace{\boldsymbol{R}\cdot\boldsymbol{R}^T}_{\boldsymbol{1}}\cdot\boldsymbol{v}^T=\boldsymbol{v}\cdot\boldsymbol{v}^T=\boldsymbol{v}^2\ .$$

This means that $\boldsymbol{b}$ and $\boldsymbol{b}^{-1}$ are indpendent of $\boldsymbol{R}$. Consequently,

$$\boldsymbol{e}=\frac{1}{2}\left(\boldsymbol{1}-\boldsymbol{b}^{-1}\right)$$

is also independent of $\boldsymbol{R}$.

Problem 2.15. Assume that the displacement field is given by the following equations:

$$u_1=a(2X_1^2+X_1X_2)\,,\qquad u_2=aX_2^2\,,\qquad u_3=0;\qquad a=10^{-4}.$$

Find the axial strains in the directions $\mathbf{i}_1$ and $\mathbf{i}_2$ at the point $P(1,1,0)$. What is the angle change between these directions?

Solution: Utilizing equation (2.36) we get

$$E_{11}=\mathbf{i}_1\cdot\mathbf{E}\cdot\mathbf{i}_1=u_{1,1}+\frac{1}{2}(u_{1,1}u_{1,1}+u_{2,1}u_{2,1}+u_{3,1}u_{3,1})=$$

$$=a\,(4X_1+X_2)+\frac{a^2}{2}\left((4X_1+X_2)^2+0+0\right),$$

$$E_{11}\big|_P=5a+12.5a^2;$$

$$\varepsilon^{o1} = \sqrt{1 + 2\mathbf{i}_1 \cdot \mathbf{E} \cdot \mathbf{i}_1} - 1 = \sqrt{1 + 2\left(5a + 12.5a^2\right)} - 1 \approx 5a + 12.5a^2$$

or

$$\varepsilon^{o1} = \sqrt{1 + 2\left(5a + 12.5a^2\right)} - 1 =$$
$$= \sqrt{1 + 2 \times \left(5 \times 10^{-4} + 12.5 \times 10^{-8}\right)} - 1 = 0.000\,5$$

and

$$\varepsilon^{o1} \approx 5 \times 10^{-4} + 12.5 \times 10^{-8} = 0.000\,500\,125 \approx 0.000\,5 = 5a.$$

We obtain in the same way

$$E_{22} = \mathbf{i}_2 \cdot \mathbf{E} \cdot \mathbf{i}_2 = u_{2,2} + \frac{1}{2}\left(u_{1,2}u_{1,2} + u_{2,2}u_{2,2} + u_{3,2}u_{3,2}\right) =$$
$$= 2aX_2 + \frac{a^2}{2}\left(X_2^2 + 4X_2^2 + 0\right),$$
$$E_{22}|_P = 2a + 2.5a^2;$$
$$\varepsilon^{o1} = \sqrt{1 + 2\mathbf{i}_2 \cdot \mathbf{E} \cdot \mathbf{i}_2} - 1 = \sqrt{1 + 2\left(2a + 2.5a^2\right)} - 1 \approx 2a + 2.5a^2$$

or

$$\varepsilon^{o2} = \sqrt{1 + 2\left(2a + 2.5a^2\right)} - 1 =$$
$$= \sqrt{1 + 2 \times \left(2 \times 10^{-4} + 2.5 \times 10^{-8}\right)} - 1 = 0.000\,200\,005$$

and

$$\varepsilon^{o1} \approx 2 \times 10^{-4} + 2.5 \times 10^{-8} = 0.000\,200\,025 \approx 0.000\,2 = 2a\,.$$

With equation (2.44) we have

$$\sin\gamma_{12} = \frac{2\mathbf{i}_1 \cdot \mathbf{E} \cdot \mathbf{i}_2}{\left(1 + \varepsilon^{o1}\right)\left(1 + \varepsilon^{o2}\right)} = \frac{2E_{12}}{\left(1 + \varepsilon^{o1}\right)\left(1 + \varepsilon^{o2}\right)},$$

where

$$E_{12} = \mathbf{i}_1 \cdot \mathbf{E} \cdot \mathbf{i}_2 = \frac{1}{2}\left(u_{1,2} + u_{2,1} + u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}\right) =$$
$$= \frac{1}{2}\left(aX_1 + 0 + a^2\left(\left(4X_1 + X_2\right)X_1 + 0 + 0\right)\right) = 0.5a + 2.5a^2\,.$$

Hence

$$\sin\gamma_{12} = \frac{E_{12}}{\left(1 + \varepsilon^{o1}\right)\left(1 + \varepsilon^{o2}\right)} = \frac{a + 5a^2}{\left(1 + \varepsilon^{o1}\right)\left(1 + \varepsilon^{o2}\right)} =$$
$$= \frac{10^{-4} + 5 \times 10^{-8}}{\left(1 + 0.000\,500\,125\right)\left(1 + 0.000\,200\,025\right)} = 0.9997\,998\,900\,89 \times 10^{-4}$$

from where

$$\sin\gamma_{12} \approx \gamma_{12} = 0.9997\,998\,900\,89 \times 10^{-4}.$$

If we take into account that $\left|\varepsilon^{o1}\right| \ll 1$, t $\left|\varepsilon^{o2}\right| \ll 1$ and neglect the quadratic term in E_{12} we obtain

$$\sin\gamma_{12} \approx \gamma_{12} = a = 10^{-4}.$$

Note that the quadratic terms can really be neglected.

B.3. Problems in Chapter 3

Problem 3.1. Given the displacement field of a continuum in spatial description:

$$u_1 = x_1 + \frac{1}{2}(x_1 + x_2)\,e^{-t} - \frac{1}{2}(x_1 - x_2)\,e^{t},$$

$$u_2 = x_2 - \frac{1}{2}(x_1 + x_2)\,e^{-t} - \frac{1}{2}(x_1 - x_2)\,e^{t}, \quad u_3 = 0.$$

Find the velocity and acceleration fields both in material (Lagrangian) and in spatial (Eulerian) descriptions.

Solution: Since $u_\ell = x_\ell - X_\ell$ we have

$$\begin{aligned} X_1 &= -\frac{1}{2}(x_1 + x_2)\,e^{-t} + \frac{1}{2}(x_1 - x_2)\,e^{t}, \\ X_2 &= -\frac{1}{2}(x_1 + x_2)\,e^{-t} - \frac{1}{2}(x_1 - x_2)\,e^{t}, \\ X_3 &= x_3 \end{aligned}$$

from where

$$\begin{bmatrix} \frac{e^t - e^{-t}}{2} & -\frac{e^t + e^{-t}}{2} \\ -\frac{e^t + e^{-t}}{2} & \frac{e^t - e^{-t}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \sinh t & -\cosh t \\ -\cosh t & \sinh t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Hence

$$x_1 = -X_1 \sinh t - X_2 \cosh t\,, \qquad x_2 = -X_1 \cosh t - X_2 \sinh t\,,$$
$$x_3 = X_3\,.$$

With the motion law

$$v_1 = \frac{\partial x_1}{\partial t} = -X_1 \cosh t - X_2 \sinh t = x_2\,,$$

$$v_2 = \frac{\partial x_2}{\partial t} = -X_1 \sinh t - X_2 \cosh t = x_1\,, \qquad v_3 = 0$$

are the velocity components in material and spatial descriptions. As regards the acceleration components we get

$$\begin{aligned} a_1 &= \frac{\partial v_1}{\partial t} = -X_1 \sinh t - X_2 \cosh t = x_1, \\ a_2 &= \frac{\partial v_2}{\partial t} = -X_1 \cosh t - X_2 \sinh t = x_2\,, \\ a_3 &= 0\,. \end{aligned}$$

Problem 3.2. Assume that

$$x_1 = X_1 e^{-t} - X_3\left(1 - e^{-t}\right), \quad x_2 = X_2 - X_3\left(e^{t} - e^{-t}\right), \quad x_3 = X_3 e^{-t}$$

is the motion law. Find the velocity and acceleration fields both in material and in spatial descriptions.

Solution: It can be checked with ease that the inverse motion law is of the form:

$$X_1 = \frac{e^t}{e^{-t}}\left(x_3 + x_1 e^{-t} - x_3 e^{-t}\right), \quad X_2 = x_2 - x_3 + x_3 e^{2t}, \quad X_3 = x_3 e^{t}.$$

With

$$X_3 = \frac{x_3}{e^{-t}} = x_3 e^{t}$$

and

$$X_1 + X_3 = \frac{e^t}{e^{-t}} \left(x_3 + x_1 e^{-t}\right)$$

we get the velocities:

$$v_1 = \frac{\partial x_1}{\partial t} = -\left(X_1 + X_3\right) e^{-t} = -\frac{1}{e^{-t}} \left(x_3 + x_1 e^{-t}\right),$$
$$v_2 = \frac{\partial x_2}{\partial t} = -X_3 \left(e^t + e^{-t}\right) = -x_3 e^t \left(e^t + e^{-t}\right),$$
$$v_3 = \frac{\partial x_3}{\partial t} = -X_3 e^{-t} = -x_3$$

and the accelerations:

$$a_1 = \frac{\partial v_1}{\partial t} = \left(X_1 + X_3\right) e^{-t} = \frac{1}{e^{-t}} \left(x_3 + x_1 e^{-t}\right),$$
$$a_2 = \frac{\partial v_2}{\partial t} = -X_3 \left(e^t - e^{-t}\right) = -x_3 e^t \left(e^t - e^{-t}\right),$$
$$a_3 = \frac{\partial v_3}{\partial t} = X_3 e^{-t} = x_3.$$

Problem 3.3. Given the velocity field of a continuum: $\mathbf{v} = \mathbf{x}/(1+t)$: prove that the motion law then takes the form $\mathbf{x} = \mathbf{X}\,(1+t)$. Determine the velocity and acceleration fields both in material (Lagrangian) and in spatial (Eulerian) descriptions.

Solution: It is obvious that

$$\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t} = \mathbf{X},$$

where $\mathbf{X} = \mathbf{x}/(1+t)$. Hence

$$\mathbf{v} = \frac{\mathbf{x}}{1+t}.$$

The last two relations are the velocities in material and spatial descriptions. The acceleration is zero:

$$\mathbf{a} = \frac{\partial^2 \mathbf{x}}{\partial t^2} = \frac{\partial^2}{\partial t^2} \mathbf{X}(1+t) = \mathbf{0}.$$

Problem 3.4. Given the velocity field for a motion in the following form:

$$v_1 = \alpha x_3, \qquad v_2 = -\beta x_3, \qquad v_3 = -\alpha x_1 + \beta x_2$$

where α and β are non zero constants. Verify that this motion is a rigid body motion. Find the spin vector.

Solution: Since

$$[l_{pq}] = [v_{p,q}] = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & -\beta \\ -\alpha & \beta & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

is skew it follows that

$$d_{pq} = 0 \qquad \text{and} \qquad l_{pq} = \Omega_{pq}.$$

Hence the motion is rigid body motion for which

$$w_1 = 2\omega_1 = 2\beta, \qquad w_2 = 2\omega_2 = 2\alpha, \qquad v_3 = 2\omega_2 = 0$$

is the spin vector.

Problem 3.5. Given the velocity field of a continuum in spatial description:

$$v_1 = -\frac{2x_1x_2x_3}{R^4}\,, \quad v_2 = \frac{x_1^2 - x_2^2}{R^4}x_1\,, \quad v_3 = \frac{x_2}{R^2}$$

where $R = \sqrt{x_1^2 + x_2^2} \neq 0$. Find the velocity gradient, the strain rate tensor, the spin tensor, the vorticity vector and the acceleration.

Solution: Making use of equation (3.6b) we obtain the velocity gradient:

$$[l_{pq}] = [v_{p,q}] = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} =$$

$$= \frac{1}{R^4}\begin{bmatrix} -\frac{2x_2x_3}{R^2}\left(x_2^2 - 3x_1^2\right) & -\frac{2x_1x_3}{R^2}\left(x_1^2 - 3x_2^2\right) & -2x_1x_2 \\ -\frac{2x_1x_3}{R^2}\left(x_1^2 - 3x_2^2\right) & \frac{2x_2x_3}{R^2}\left(x_2^2 - 3x_1^2\right) & x_1^2 - x_2^2 \\ -2x_1x_2 & x_1^2 - x_2^2 & 0 \end{bmatrix}$$

Since

$$l_{pq} = l_{qp}$$

it follows that the velocity gradient coincides with the strain rate tensor:

$$d_{pq} = l_{pq}$$

Hence

$$\Omega_{pq} = \frac{1}{2}(l_{pq} - l_{qp}) = 0 \qquad \text{and} \qquad \omega_r = -\frac{1}{2}e_{pqr}\Omega_{pq} = 0$$

which means that the spin tensor and the vorticity vector $w_r = 2\omega_r$ vanish. Then the motion is said to be irrotational.

Problem 3.6. Assume that the velocity field is the gradient of a potential function ϕ, i.e., $\mathbf{v} = \phi\nabla$. Prove that the right side of equation

$$\frac{\mathrm{D}}{\mathrm{D}t}(\mathbf{v}) = (\mathbf{v})^{\cdot} = \left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\nabla$$

is really the acceleration field.

Solution: Substituting $\mathbf{v} = \phi\nabla$ into equation (3.58a) yields

$$\mathbf{a} = \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = (\mathbf{v})^{\cdot} = (\mathbf{v}\circ\nabla)\cdot\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t} = \overset{\downarrow}{\mathbf{v}}(\mathbf{v}\cdot\nabla) + \frac{\partial\phi}{\partial t}\nabla = \left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\nabla\,.$$

Problem 3.7. Given the velocity field of a continuum in spatial description:

$$v_1 = \frac{f(R)}{R}x_2\,, \quad v_2 = -\frac{f(R)}{R}x_1\,, \quad v_3 = 0$$

where $R = \sqrt{x_1^2 + x_2^2} \neq 0$. Prove that this motion is volume preserving. Show, in addition to this, that the spin vector (or the angular velocity vector) vanishes if $f(R) = -1/R$.

Solution: It can be checked with ease that

$$[l_{pq}] = [v_{p,q}] = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} =$$

$$= \begin{bmatrix} \left(\frac{\mathrm{d}f(R)}{R\mathrm{d}R} - \frac{f(R)}{R^2}\right)\frac{x_1 x_2}{R} & \left(\frac{\mathrm{d}f(R)}{R\mathrm{d}R} - \frac{f(R)}{R^2}\right)\frac{x_2^2}{R} + \frac{f(R)}{R} & 0 \\ -\left(\frac{\mathrm{d}f(R)}{R\mathrm{d}R} - \frac{f(R)}{R^2}\right)\frac{x_1^2}{R} - \frac{f(R)}{R} & -\left(\frac{\mathrm{d}f(R)}{R\mathrm{d}R} - \frac{f(R)}{R^2}\right)\frac{x_1 x_2}{R} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[d_{pq}] = \frac{1}{2}\left[l_{pq} + l_{qp}\right] = \begin{bmatrix} \left(\frac{\mathrm{d}f(R)}{R\mathrm{d}R} - \frac{f(R)}{R^2}\right)\frac{x_1 x_2}{R} & \frac{1}{2}\left(\frac{\mathrm{d}f(R)}{R\mathrm{d}R} - \frac{f(R)}{R^2}\right)\frac{x_2^2 - x_1^2}{R} & 0 \\ \frac{1}{2}\left(\frac{\mathrm{d}f(R)}{R\mathrm{d}R} - \frac{f(R)}{R^2}\right)\frac{x_2^2 - x_1^2}{R} & -\left(\frac{\mathrm{d}f(R)}{R\mathrm{d}R} - \frac{f(R)}{R^2}\right)\frac{x_1 x_2}{R} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$[\Omega_{pq}] = \frac{1}{2}\left[l_{pq} - l_{qp}\right] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\mathrm{d}f(R)}{\mathrm{d}R} - \frac{f(R)}{R}\right) + \frac{f(R)}{R} & 0 \\ \frac{1}{2}\left(\frac{\mathrm{d}f(R)}{\mathrm{d}R} - \frac{f(R)}{R}\right) + \frac{f(R)}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -\omega_3 & 0 \\ \omega_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from where

$$w_3 = 2\omega_3 = -\frac{1}{2}\left(\frac{\mathrm{d}f(R)}{\mathrm{d}R} - \frac{f(R)}{R}\right) - \frac{f(R)}{R}$$

is the spin vector.

Since $d_I = d_{11} + d_{22} + d_{33} = 0$ equation (3.33) yields

$$(\mathrm{d}V)^{\boldsymbol{\cdot}} = d_I \, \mathrm{d}V = 0$$

which means that the motion is really volume preserving. For $f(R) = -1/R$ the spin vector is zero, i.e., the motion is irrotational.

B.4. Problems in Chapter 4

Problem 4.1. Given the displacement field of a solid body by equation (4.79). Determine the matrices $[\varepsilon_{k\ell}]$ and $[\Psi_{k\ell}]$. With the displacements at the points

$$\mathbf{X}_P = -20\mathbf{i}_1 + 30\mathbf{i}_2 + 40\mathbf{i}_3 \quad [\mathrm{mm}]\,, \qquad \mathbf{X}_Q = \mathbf{X}_P + \mathbf{i}_1 \quad [\mathrm{mm}]\,.$$

compare the value $\Delta\mathbf{u} = \mathbf{u}_Q - \mathbf{u}_P$ calculated using the exact solution for $\mathbf{u}$ and the approximation

$$\Delta\mathbf{u} \approx (\mathbf{u} \circ \nabla)|_{P^\circ} \cdot \Delta\mathbf{X}\,, \qquad \Delta\mathbf{X} = \mathbf{X}_Q - \mathbf{X}_P = \mathbf{i}_1 \quad [\mathrm{mm}]\,.$$

Solution: Making use of the derivatives

$$\mathbf{u}_{,1} = \frac{\partial \mathbf{u}}{\partial X_1} = CX_2^2\mathbf{i}_1 + 2CX_1X_3\mathbf{i}_3\,, \quad \mathbf{u}_{,2} = \frac{\partial \mathbf{u}}{\partial X_2} = 2CX_1X_2\mathbf{i}_1 + CX_3^2\mathbf{i}_2\,,$$
$$\mathbf{u}_{,3} = \frac{\partial \mathbf{u}}{\partial X_3} = 2CX_2X_3\mathbf{i}_2 + CX_1^2\mathbf{i}_3$$

we get the displacement gradient and its value at the point P:

$$[u_{k,l}] = C\begin{bmatrix} X_2^2 & 2X_1X_2 & 0 \\ 0 & X_3^2 & 2X_2X_3 \\ 2X_1X_3 & 0 & X_1^2 \end{bmatrix} \quad \text{and} \quad [u_{k,l}(P)] = \begin{bmatrix} 9 & -12 & 0 \\ 0 & 16 & 24 \\ -16 & 0 & 4 \end{bmatrix} 10^{-3}\,.$$

On the basis of (4.17) we can write

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{u}\circ\nabla + \nabla\circ\mathbf{u})\,, \qquad [\varepsilon_{k\ell}] = \left[u_{(k\ell)}\right] = \left[\frac{1}{2}(u_{k,\ell} + u_{\ell,k})\right]$$

from where

$$[\varepsilon_{k\ell}] = C\begin{bmatrix} X_2^2 & X_1X_2 & X_1X_3 \\ X_1X_2 & X_3^2 & X_2X_3 \\ X_1X_3 & X_2X_3 & X_1^2 \end{bmatrix} \quad \text{and} \quad [\varepsilon_{k\ell}(P)] = \begin{bmatrix} 9 & -6 & -8 \\ -6 & 16 & 12 \\ -8 & 12 & 4 \end{bmatrix} 10^{-3}$$

are the symmetric part of the displacement gradient $u_{k,\ell}$ and its value at the point P. For the skew part we get in the same way that

$$[\Psi_{k\ell}] = \left[u_{[k\ell]}\right] = \left[\frac{1}{2}(u_{k,\ell} - u_{\ell,k})\right] = C\begin{bmatrix} 0 & X_1X_2 & -X_1X_3 \\ -X_1X_2 & 0 & X_2X_3 \\ X_1X_3 & -X_2X_3 & 0 \end{bmatrix}\,,$$
$$[\Psi_{k\ell}(P)] = \begin{bmatrix} 0 & -6 & 8 \\ 6 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix} 10^{-3}.$$

Recalling $(4.21b)_2$

$$[\Psi_{k\ell}] = \begin{bmatrix} 0 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 0 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 0 \end{bmatrix}$$

is the matrix of the rotation tensor in terms of the rotation components. Hence

$$\boldsymbol{\varphi} = -CX_2X_3\,\mathbf{i}_1 - CX_1X_3\,\mathbf{i}_2 - CX_2X_3\,\mathbf{i}_3 \qquad \text{and} \qquad \boldsymbol{\varphi}(P) = (-12\mathbf{i}_1 + 8\mathbf{i}_2 + 6\mathbf{i}_3)\,10^{-3}$$

are the rotation vector and its value at the point P. The displacements at the points P and Q are obtained by simple substitutions:

$$\begin{aligned}\mathbf{u}(P) &= -10^{-5}\times 20\times 30^2\mathbf{i}_1 + 10^{-5}\times 30\times 40^2\mathbf{i}_2 + 10^{-5}\times 40\times 20^2\mathbf{i}_3 = \\ &= -0.18\mathbf{i}_1 + 0.48\mathbf{i}_2 + 0.16\mathbf{i}_3\ [\text{mm}],\\ \mathbf{u}(Q) &= -10^{-5}\times 19\times 30^2\mathbf{i}_1 + 10^{-5}\times 30\times 40^2\mathbf{i}_2 + 10^{-5}\times 40\times 19^2\mathbf{i}_3 = \\ &= -0.171\mathbf{i}_1 + 0.48\mathbf{i}_2 + 0.1444\mathbf{i}_3\ [\text{mm}].\end{aligned}$$

On the basis of equation (4.20a) the approximate value of the displacement vector at Q is given by

$$\mathbf{u}(Q) = \mathbf{u}(P) + \mathbf{u}\circ\nabla|_P\cdot \mathrm{d}\mathbf{X}\,.$$

Consequently, $(\mathrm{d}X_\ell = X_\ell(Q) - X_\ell(P))$ we get

$$[u_k(Q)] \simeq [u_k(P)] + [u_{k,\ell}(P)]\,[\mathrm{d}X_\ell] = \begin{bmatrix} -0.18 \\ 0.48 \\ 0.16 \end{bmatrix} + 10^{-3} \begin{bmatrix} 9 & -12 & 0 \\ 0 & 16 & 24 \\ -16 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$$
$$= \begin{bmatrix} -0.18 \\ 0.48 \\ 0.16 \end{bmatrix} + \begin{bmatrix} 0.009 \\ 0 \\ -0.016 \end{bmatrix} = \begin{bmatrix} -0.171 \\ 0.48 \\ 0.144 \end{bmatrix} [\mathrm{mm}]$$

is the approximate value of the displacement at Q. The difference between the exact and approximate values can now be calculated:

$$[\Delta u_k(Q)] = u_k(Q)^{(\mathrm{exact})} - u_k(Q)^{(\mathrm{approximative})} = \begin{bmatrix} 0 \\ 0 \\ 0.0004 \end{bmatrix} [\mathrm{mm}] .$$

Problem 4.2. Show that the diagonal and off-diagonal components of the incompatibility tensor are given by equations (4.42).

Solution: Equation (4.41) defines the tensor of incompatibility:

$$\eta_{r\ell} = e_{qpr} e_{ks\ell} \varepsilon_{ps,qk} .$$

If we utilize the properties of the permutation symbol, the symmetry of the strain tensor and the interchangeability of the derivations we get:

$$\eta_{11} = e_{qp1} e_{ks1} \varepsilon_{ps,qk} =$$
$$= e_{231} e_{231} \varepsilon_{33,22} + e_{321} e_{321} \varepsilon_{22,33} + \underbrace{e_{231} e_{321}}_{-1} \varepsilon_{32,23} + \underbrace{e_{321} e_{231}}_{-1} \varepsilon_{23,32}$$

from where

$$\eta_{11} = \varepsilon_{22,33} + \varepsilon_{33,22} - 2\varepsilon_{23,23} .$$

We can proceed in the same way:

$$\eta_{22} = e_{qp2} e_{ks2} \varepsilon_{ps,qk} =$$
$$= e_{132} e_{132} \varepsilon_{33,11} + e_{312} e_{312} \varepsilon_{11,33} + \underbrace{e_{132} e_{312}}_{-1} \varepsilon_{31,13} + \underbrace{e_{312} e_{132}}_{-1} \varepsilon_{13,31} =$$
$$= \varepsilon_{33,11} + \varepsilon_{11,33} - 2\varepsilon_{31,31} ,$$

$$\eta_{33} = e_{qp3} e_{ks3} \varepsilon_{ps,qk} =$$
$$e_{123} e_{123} \varepsilon_{22,11} + e_{213} e_{213} \varepsilon_{11,22} + \underbrace{e_{123} e_{213}}_{-1} \varepsilon_{21,12} + \underbrace{e_{213} e_{123}}_{-1} \varepsilon_{12,21} =$$
$$= \varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} .$$

For the off diagonal elements of the incompatibility tensor we obtain

$$\eta_{12} = e_{qp1} e_{ks2} \varepsilon_{ps,qk} =$$
$$= e_{231} e_{312} \varepsilon_{31,23} + e_{321} e_{132} \varepsilon_{23,31} + \underbrace{e_{231} e_{132}}_{-1} \varepsilon_{33,21} + \underbrace{e_{321} e_{312}}_{-1} \varepsilon_{21,33} =$$
$$= \varepsilon_{31,23} + \varepsilon_{23,31} - \varepsilon_{33,21} - \varepsilon_{21,33} = \left(\varepsilon_{13,2} + \varepsilon_{23,1} - \varepsilon_{12,3}\right)_{,3} - \varepsilon_{33,12} ,$$

$$\eta_{23} = e_{qp2}e_{ks3}\varepsilon_{ps,qk} =$$
$$= e_{312}e_{123}\varepsilon_{12,31} + e_{132}e_{213}\varepsilon_{31,12} + \underbrace{e_{312}e_{213}}_{-1}\varepsilon_{11,32} + \underbrace{e_{132}e_{123}}_{-1}\varepsilon_{32,11} =$$
$$= \varepsilon_{12,31} + \varepsilon_{31,12} - \varepsilon_{11,32} - \varepsilon_{32,11} = \left(\varepsilon_{21,3} + \varepsilon_{31,2} - \varepsilon_{23,1}\right)_{,1} - \varepsilon_{11,23}\,,$$

$$\eta_{31} = e_{qp3}e_{ks1}\varepsilon_{ps,qk} =$$
$$= e_{123}e_{231}\varepsilon_{23,12} + e_{213}e_{321}\varepsilon_{12,23} + \underbrace{e_{123}e_{321}}_{-1}\varepsilon_{22,13} + \underbrace{e_{213}e_{231}}_{-1}\varepsilon_{13,22} =$$
$$= \varepsilon_{23,12} + \varepsilon_{12,23} - \varepsilon_{22,13} - \varepsilon_{13,22} = \left(\varepsilon_{32,1} + \varepsilon_{12,3} - \varepsilon_{31,2}\right)_{,2} - \varepsilon_{22,31}\,.$$

Problem 4.3. Assume that (a) $\varepsilon_{\kappa\lambda}$ is independent of X_3 and $\varepsilon_{\kappa 3} = \varepsilon_{3\kappa} = \varepsilon_{33} = 0$. Find the compatibility equations for this strain tensor and clarify the conditions under which the strain components

$$\varepsilon_{11} = k(X_1^2 - X_2^2)\,, \qquad \varepsilon_{12} = \varepsilon_{21} = \ell X_1 X_2\,, \qquad \varepsilon_{22} = k X_1 X_2\,, \\ \varepsilon_{\kappa 3} = \varepsilon_{3\kappa} = \varepsilon_{33} = 0 \tag{B.4.9}$$

are compatible if k and ℓ are arbitrary not zero constants.

Solution: Since (a) $\varepsilon_{\kappa\lambda}$ is independent of X_3 and (b) $\varepsilon_{\kappa 3} = \varepsilon_{3\kappa} = \varepsilon_{33} = 0$ it follows from equations (4.42) that the not identically zero component of the incompatibility tensor $\boldsymbol{\eta}$ is

$$\eta_{33} = \varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12}\,.$$

Since

$$\varepsilon_{11,22} = -2k\,, \qquad \varepsilon_{22,11} = 0\,, \qquad \varepsilon_{12,12} = \ell$$

the compatibility condition $\eta_{33} = 0$ yields

$$\ell + k = 0\,.$$

This equation shows that the not identically zero strain components $\varepsilon_{\kappa\lambda}$ are, in general, not compatible except the case when $k = -\ell$. Then

$$\varepsilon_{11} = \ell(X_2^2 - X_1^2)\,, \qquad \varepsilon_{12} = \varepsilon_{21} = \ell X_1 X_2\,, \qquad \varepsilon_{22} = -\ell X_1 X_2\,, \\ \varepsilon_{\kappa 3} = \varepsilon_{3\kappa} = \varepsilon_{33} = 0\,. \tag{B.4.10}$$

We remark that the strain state for which conditions (a) and (b) are satisfied is called plain strain.

Problem 4.4. The strain field in Problem 4.3 is compatible if $k = -\ell$. Find the rotation field.

Solution: We apply the first Cesaro formula (4.68) to find the angle of rotation. Since the strain components are such that (a) $\varepsilon_{\kappa\lambda}$ is independent of X_3 and $\varepsilon_{\kappa 3} = \varepsilon_{3\kappa} = \varepsilon_{33} = 0$ the first Cesaro formula

$$\varphi_r\,(P) = \varphi_r\,(B) + \int_g e_{mkr}\varepsilon_{k\ell,m}\tau_\ell\,\mathrm{d}s$$

can be rewritten in the following form

$$\varphi_3\,(P) = \varphi_3\,(B) + \int_g e_{\mu\kappa 3}\varepsilon_{\kappa\lambda,\mu}\tau_\lambda\,\mathrm{d}s\,,$$

where

$$\begin{aligned} &\varepsilon_{11,1} = -2\ell X_1, \quad \varepsilon_{11,2} = 2\ell X_2, \quad \varepsilon_{12,1} = \ell X_2, \\ &\varepsilon_{12,2} = \ell X_1, \qquad \varepsilon_{22,1} = -\ell X_2, \quad \varepsilon_{22,2} = -\ell X_1. \end{aligned} \tag{B.4.11}$$

The curve g is selected in the following manner: (a) the point B is the origin, (b) g_1 is a horizontal line segment between the points $(0,0)$ and $(X_1, 0)$, i.e., the part of the axis X_1 from the origin to the point with abscissa X_1, (c) g_2 is a vertical line segment between the points $(X_1, 0)$ and (X_1, X_2) and $g = g_1 \cup g_2$. Then

$$\begin{aligned} \varphi_3(P) &= \varphi_3(O) + \int_g e_{\mu\kappa 3}\varepsilon_{\kappa\lambda,\mu}\tau_\lambda \,\mathrm{d}s = \varphi_3(O) + \int_g (\varepsilon_{2\lambda,1} - \varepsilon_{1\lambda,2})\,\tau_\lambda \,\mathrm{d}s = \\ &= \varphi_3(O) + \int_0^{X_1} (\varepsilon_{21,1} - \varepsilon_{11,2})\,\tau_1 \,\mathrm{d}X_1 + \int_0^{X_2} (\varepsilon_{22,1} - \varepsilon_{12,2})\,\tau_2 \,\mathrm{d}X_2 = \underset{\tau_1=\tau_2=1}{\uparrow} = \\ &= \varphi_3(O) + \ell \int_0^{X_1} (X_2 + 2X_2)|_{X_2=0} \,\mathrm{d}X_1 - \ell \int_0^{X_2} (X_2 + X_1)\,\mathrm{d}X_2 = \\ &= \varphi_3(O) - \ell\left(\frac{X_2^2}{2} + X_1 X_2\right) \end{aligned}$$

is the rotation field.

Problem 4.5. Find the displacement components for the strain field in Problem 4.3.

Solution: We apply the second Cesaro formula to find the unknown displacement components. Since the strain state given by equation (4.81) is a plane strain it follows that the second Cesaro formula (4.69) is of the form

$$\begin{aligned} u_\nu(P) &= u_\nu(B) + e_{3\chi\nu}\varphi_3(B)\,(X_\chi(P) - X_q(B)) + \\ &\int_g \{\varepsilon_{\nu\lambda} + (X_\chi(P) - X_\chi)\,(\delta_{\nu\mu}\delta_{\chi\kappa} - \delta_{\nu\kappa}\delta_{\chi\mu})\;\varepsilon_{\kappa\lambda,\mu}\}\,\tau_\lambda \,\mathrm{d}s\,. \end{aligned}$$

The curve g is selected in the same manner as for Problem 4.4. Thus

$$\begin{aligned} u_\nu(P) &= u_\nu(O) + e_{3\chi\nu}\varphi_3(O)\,X_\chi(P) + \\ &+ \int_g \{\varepsilon_{\nu\lambda} + [X_\chi(P) - X_\chi]\,(\delta_{\nu\mu}\delta_{\chi\kappa} - \delta_{\nu\kappa}\delta_{\chi\mu})\;\varepsilon_{\kappa\lambda,\mu}\}\,\tau_\lambda \,\mathrm{d}s\,. \end{aligned}$$

If $\nu = 1$ we have

$$\begin{aligned} u_1(P) &= u_1(O) - \varphi_3(O)\,X_1(P) + \\ &+ \int_g \{\varepsilon_{1\lambda} + (X_\chi(P) - X_\chi)\,\varepsilon_{\chi\lambda,1} - (X_\chi(P) - X_\chi)\,\varepsilon_{1\lambda,\chi}\}\,\tau_\lambda \,\mathrm{d}s\,. \end{aligned}$$

Here

$$\begin{aligned} &\int_g \{\varepsilon_{1\lambda} + (X_\chi(P) - X_\chi)\,\varepsilon_{\chi\lambda,1} - (X_\chi(P) - X_\chi)\,\varepsilon_{1\lambda,\chi}\}\,\tau_\lambda \,\mathrm{d}s = \\ &\int_0^{X_1} \{\varepsilon_{11} + 1\,(X_\chi(P) - X_\chi)\,\varepsilon_{\chi 1,1} - (X_\chi(P) - X_\chi)\,\varepsilon_{11,\chi}\}|_{X_2=0} \,\mathrm{d}X_1 + \\ &+ \int_0^{X_2} \{\varepsilon_{12} + (X_\chi(P) - X_\chi)\,\varepsilon_{\chi 2,1} - (X_\chi(P) - X_\chi)\,\varepsilon_{12,\chi}\}\,\mathrm{d}X_2\,, \end{aligned}$$

in which the two line integrals on the right side can be calculated by using relations (B.4.10) and (B.4.11). We get

$$\int_0^{X_1} \{\varepsilon_{11} + (X_1(P) - X_1)\,\varepsilon_{11,1} + (X_2(P) - X_2)\,\varepsilon_{21,1} - \\ - (X_1(P) - X_1)\,\varepsilon_{11,1} - (X_2(P) - X_2)\,\varepsilon_{11,2}\}\,|_{X_2=0}\,\mathrm{d}X_1 =$$

$$= \int_0^{X_1} \{\varepsilon_{11} + (X_2(P) - X_2)\,\varepsilon_{21,1} - (X_2(P) - X_2)\,\varepsilon_{11,2}\}|_{X_2=0}\,\mathrm{d}X_1 =$$

$$= \ell \int_0^{X_1} \left\{-X_1^2 - (X_1(P) - X_1)\,X_2 + 2\,(X_1(P) - X_1)\,X_2\,\right\}\Big|_{X_2=0}\,\mathrm{d}X_1 = -\ell\frac{X_1^3}{3}$$

and

$$\int_0^{X_2} \{\varepsilon_{12} + (X_\chi(P) - X_\chi)\,\varepsilon_{\chi 2,1} - (X_\chi(P) - X_\chi)\,\varepsilon_{12,\chi}\}\,\mathrm{d}X_2 =$$

$$= \int_0^{X_2} \{\varepsilon_{12} + (X_1(P) - X_1)\,\varepsilon_{12,1} + (X_2(P) - X_2)\,\varepsilon_{22,1} - \\ - (X_1(P) - X_1)\,\varepsilon_{12,1} - (X_2(P) - X_2)\,\varepsilon_{12,2}\}\,\mathrm{d}X_2 =$$

$$= \ell \int_0^{X_2} \{X_1X_2 + (X_1(P) - X_1)\,X_2 - (X_2(P) - X_2)\,X_2 - \\ - (X_1(P) - X_1)\,X_2 + (X_2(P) - X_2)\,X_1\}\,\mathrm{d}X_2 =$$

$$= \ell \int_0^{X_2} \{-X_2(P)\,X_2 + X_2X_2 + X_2(P)\,X_1\}\,\mathrm{d}X_2 =$$

$$= \ell\left(-\frac{X_2^3}{2} + \frac{X_2^3}{3} + X_1X_2^2\right) = -\frac{\ell}{6}\left(-6X_1X_2^2 + X_2^3\right)\,,$$

where it has been taken into account that $X_1(P) = X_1$ in the second integral. Hence

$$u_1(P) = u_1(O) - \varphi_3(O)\,X_1(P) - \frac{\ell}{6}\left(2X_1^3 - 6X_1X_2^2 + X_2^3\right)\,.$$

As regards u_2 repeating the previous line of thought we get

$$u_2(P) = u_2(O) + \varphi_3(O)\,X_1(P) + \\ + \int_0^{X_1} \{\varepsilon_{21} + (X_\chi(P) - X_\chi)\,\varepsilon_{\chi 1,2} - (X_\chi(P) - X_\chi)\,\varepsilon_{21,\chi}\,\}|_{X_2=0}\,\mathrm{d}X_1 + \\ + \int_0^{X_2} \{\varepsilon_{22} + (X_\chi(P) - X_\chi)\,\varepsilon_{\chi 2,2} - (X_\chi(P) - X_\chi)\,\varepsilon_{22,\chi}\,\}\,\mathrm{d}X_2\,,$$

where the two line integrals are given by the following equations:

$$\int_0^{X_1} \{\varepsilon_{21} + (X_1(P) - X_1)\,\varepsilon_{11,2} + (X_2(P) - X_2)\,\varepsilon_{21,2} - \\ - (X_1(P) - X_1)\,\varepsilon_{21,1} - (X_2(P) - X_2)\,\varepsilon_{21,2}\,\}\,|_{X_2=0}\,\mathrm{d}X_1 =$$

$$= \ell \int_0^{X_1} \{X_1X_2 + 2\,(X_1(P) - X_1)\,X_2 + (X_2(P) - X_2)\,X_1 - \\ - (X_1(P) - X_1)\,X_2 - (X_2(P) - X_2)\,X_1\,\}\,|_{X_2=0}\,\mathrm{d}X_1 =$$

$$= \ell \int_0^{X_1} \{X_1X_2 + (X_2(P) - X_2)\,X_1 - (X_1(P) - X_1)\,X_2 - \\ - (X_2(P) - X_2)\,X_1\,\}\,|_{X_2=0}\,\mathrm{d}X_1 = 0$$

and

$$\int_0^{X_2} \{\varepsilon_{22} + (X_1(P) - X_1)\,\varepsilon_{12,2} + (X_2(P) - X_2)\,\varepsilon_{22,2} - \\ - (X_1(P) - X_1)\,\varepsilon_{22,1} - (X_2(P) - X_2)\,\varepsilon_{22,2}\}\,\mathrm{d}X_2 =$$

$$= \ell \int_0^{X_2} \{-X_1X_2 + (X_1(P) - X_1)\,X_1 - (X_2(P) - X_2)\,X_1 + \\ + (X_1(P) - X_1)\,X_2 + (X_2(P) - X_2)\,X_1\}\,\mathrm{d}X_2 =$$

$$= \ell \int_0^{X_2} \{-X_1X_2 + (X_1(P) - X_1)\,X_1 + (X_1(P) - X_1)\,X_2\}\,\mathrm{d}X_2 =$$

$$= \ell \int_0^{X_2} \{-2X_1X_2 + (X_1(P) - X_1)\,X_1 + X_1(P)\,X_2\,\}\,\mathrm{d}X_2 = -\ell\frac{X_1X_2^2}{2}\,.$$

Thus

$$u_2(P) = u_2(O) + \varphi_3(O)\,X_1(P) - \ell\frac{X_1X_2^2}{2}\,.$$

B.5. Problems in Chapter 5

Problem 5.1. Given the matrix of the Cauchy stress tensor at the point P of the current configuration of the body. The unit normal to a plane which passes through the point P is denoted by $\mathbf{n}$:

$$\underline{\mathbf{T}} = \begin{bmatrix} 58.4 & 0.0 & -28.8 \\ 0.0 & -40.0 & 0.0 \\ -28.8 & 0.0 & 41.6 \end{bmatrix} [\mathrm{MPa}]\,, \qquad \mathbf{n} = 0.7, \mathbf{i}_1 + 0.1, \mathbf{i}_2 + \frac{\sqrt{2}}{2}\,\mathbf{e}_3\,.$$

Determine (a) the principal stresses and principal directions and (b) the normal stress $\sigma^{(n)}$ as well as the shearing stresses $\tau^{(n)}$ acting on the plane with normal $\mathbf{n}$.

Solution: It is not too difficult to check – we remind the reader of Exercise 1.7 – that the principal stresses are given by:

$$\sigma_1 = 80.0\,, \qquad \sigma_2 = 20.0\,, \qquad \sigma_3 = -40.0 \quad \left[\mathrm{N/mm^2}\right].$$

The corresponding eigenvectors are

$$\mathbf{n}_1 = -0.8\mathbf{i}_1 + 0.6\mathbf{i}_3\,, \qquad \mathbf{n}_2 = \mathbf{i}_2\,, \qquad \mathbf{n}_3 = -0.6\mathbf{i}_1 - 0.8\mathbf{i}_3\,.$$

As regards the stress vector and the normal stress on the plane with normal $\mathbf{n}$ we have

$$\underline{\mathbf{t}}^{(n)} = \underline{\mathbf{T}}\,\underline{\mathbf{n}} = \begin{bmatrix} 58.4 & 0.0 & -28.8 \\ 0.0 & -40.0 & 0.0 \\ -28.8 & 0.0 & 41.6 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.1 \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 20.515 \\ -4.0 \\ 9.256 \end{bmatrix} \left[\mathrm{N/mm^2}\right],$$

$$\sigma^{(n)} = \mathbf{n}\cdot\mathbf{t}^{(n)} = \begin{bmatrix} 0.7 & 0.1 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 20.515 \\ -4.0 \\ 9.256 \end{bmatrix} = 20.50 \left[\mathrm{N/mm^2}\right].$$

With these quantities we can calculate the shear stress by utilizing equation (5.17b):

$$\underline{\boldsymbol{\tau}}^{(n)} = \underline{\mathbf{t}}^{(n)} - \sigma^{(n)}\,\underline{\mathbf{n}} = \begin{bmatrix} 20.515 \\ -4.0 \\ 9.256 \end{bmatrix} - 20.505 \begin{bmatrix} 0.7 \\ 0.1 \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 6.162 \\ -6.051 \\ -5.243 \end{bmatrix} \left[\mathrm{N/mm^2}\right].$$

Problem 5.2. Consider a circular cylinder of radius R and assume that the axis of the cylinder coincides with the coordinate axis x_3. The Cauchy stresses in the cylinder are given by

$$t_{13} = t_{31} = -\mu\vartheta x_2\,, \qquad t_{23} = t_{32} = \mu\vartheta x_1\,,$$
$$t_{11} = t_{22} = t_{33} = t_{12} = t_{21} = 0\,,$$

where μ is a positive constant and ϑ is constant. Show that the outer surface of the cylinder is stress free.

Solution: The coordinates of a point on the outer surface is denoted by $x_1(R)$ and $x_2(R)$. x_3 is arbitrary. It is obvious that they satisfy the relation $x_1^2(R)+x_2^2(R) = R^2$. The outward unit normal

$$\mathbf{n} = \frac{x_1(R)}{R}\mathbf{i}_1 + \frac{x_2(R)}{R}\mathbf{i}_2$$

on the outer surface and the considered stress state are independent of x_3. Since the stress vector

$$\underline{\mathbf{t}}^{(n)} = \underline{\mathbf{T}}\,\underline{\mathbf{n}} = \frac{\mu\vartheta}{R}\begin{bmatrix} 0 & 0 & -x_2(R) \\ 0 & 0 & x_1(R) \\ -x_2(R) & x_1(R) & 0 \end{bmatrix}\begin{bmatrix} x_1(R) \\ x_2(R) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

on the outer surface of the cylinder is zero it follows that the outer surface is stress free (unloaded).

Problem 5.3. Given the deformation of a continuum

$$\mathbf{x} = aX_1\mathbf{i}_1 - bX_2\mathbf{i}_2 + cX_3\mathbf{i}_3\,,$$

where a, b and c are non zero constants. Assume that the Cauchy stress tensor is known:

$$[t_{k\ell}] = \begin{bmatrix} t_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad [t_{k\ell}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad [t_{k\ell}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t_{33} \end{bmatrix},$$

where $t_{11} = t_{22} = t_{33} = \sigma_\circ =$ constant. Find the first and second Piola-Kirchoff stress tensors for each $[t_{k\ell}]$.

Solution: It can be checked with ease that

$$\underline{\mathbf{F}} = \begin{bmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & c \end{bmatrix}, \qquad \underline{\mathbf{F}}^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & -1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}, \qquad J = -abc.$$

Making use of equations (5.24) and (5.24) from the first Cauchy stress tensor we get the first and second Piola-Kirchoff stress tensors:

$$\underline{\mathbf{P}} = J\,\underline{\mathbf{t}}\,\underline{\mathbf{F}}^{-1} = -abc\begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/a & 0 & 0 \\ 0 & -1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix} = -bc\begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\underline{\mathbf{S}} = J\,\underline{\mathbf{F}}^{-T}\,\underline{\mathbf{t}}\,\underline{\mathbf{F}}^{-1} =$$
$$= -abc\begin{bmatrix} 1/a & 0 & 0 \\ 0 & -1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}\begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/a & 0 & 0 \\ 0 & -1/b & 0 \\ 0 & 0 & 1/cc \end{bmatrix} =$$
$$= -bc\begin{bmatrix} \sigma_0/a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For the other two Cauchy stress tensors the procedure to be followed is the same.

Problem 5.4. Given the matrix of the Cauchy stress tensor in spatial description:

$$\underline{\mathbf{t}} = \begin{bmatrix} 0 & 0 & ax_2 \\ 0 & 0 & -bx_3 \\ ax_2 & -bx_3 & 0 \end{bmatrix}.$$

Assume that the stresses are considered at the point P with coordinates $x_1 = 0$, $x_2 = b^2$ and $x_3 = a$. Find (a) the three scalar invariants, (b) the principal stresses and principal directions and then (c) determine the maximum shear stress and the normal to the plane on which it acts.

Solution: The matrix of the stress tensor at the point P is given by

$$\underline{\mathbf{T}} = \begin{bmatrix} 0 & 0 & ab^2 \\ 0 & 0 & -ab \\ ab^2 & -ab & 0 \end{bmatrix}.$$

If we follow the solution steps of Exercise 5.4 we shall find that

$$\sigma_1 = ab\sqrt{1+b^2}\,, \qquad \sigma_2 = 0\,, \qquad \sigma_3 = -ab\sqrt{1+b^2}$$

are the three principal stresses. As regards the eigenvectors we get

$$\mathbf{n}_1 = \frac{1}{\sqrt{2\left(1+b^2\right)}}\left(b\mathbf{i}_1 - \mathbf{i}_2 + \sqrt{1+b^2}\mathbf{i}_3\right), \qquad \mathbf{n}_2 = \frac{1}{\sqrt{1+b^2}}\left(\mathbf{i}_1 + b\mathbf{i}_2\right),$$

$$\mathbf{n}_3 = \frac{1}{\sqrt{2\left(1+b^2\right)}}\left(b\mathbf{i}_1 - \mathbf{i}_2 - \sqrt{1+b^2}\mathbf{i}_3\right).$$

Making use of equations (5.55) we can determine the maximum shear stress and the outward unit normal associated to it:

$$\tau^{(n)}_{\max} = \frac{1}{2}|\sigma_1 - \sigma_3| = \left|ab\sqrt{1+b^2}\right|,$$

$$\mathbf{n} = \pm\frac{1}{\sqrt{2}}(\mathbf{n}_1 - \mathbf{n}_3) = \mathbf{i}_3\,.$$

Problem 5.5. Let $\mathbf{n}$ and $\hat{\mathbf{n}}$ the normal to two different surface elements at the point P in the current configuration of the body. The stress vectors on these surface elements are denoted by $\mathbf{t}^{(n)}$ and $\mathbf{t}^{(\hat{n})}$. Prove that the relation

$$\mathbf{n}\cdot\mathbf{t}^{(n)} = \hat{\mathbf{n}}\cdot\mathbf{t}^{(\hat{n})}$$

holds if and only if the Cauchy stress tensor is symmetric.

Solution: Recalling equation (5.14) we may write

$$\mathbf{n}\cdot\boldsymbol{t}\cdot\hat{\mathbf{n}} - \hat{\mathbf{n}}\cdot\boldsymbol{t}\cdot\mathbf{n} = \mathbf{n}\cdot\boldsymbol{t}\cdot\hat{\mathbf{n}} - \mathbf{n}\cdot\boldsymbol{t}^T\cdot\hat{\mathbf{n}} = \mathbf{n}\cdot\left(\boldsymbol{t}-\boldsymbol{t}^T\right)\cdot\hat{\mathbf{n}} = 0\,.$$

It is obvious that the difference can be zero for arbitrary $\mathbf{n}$ and $\hat{\mathbf{n}}$ if $\boldsymbol{t} = \boldsymbol{t}^T$. That was to be proved.

Problem 5.6. Prove the following equalities:

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\cdot\boldsymbol{\tau}\cdot\boldsymbol{F}^{-T}\,, \qquad S_{AB} = F^{-1}_{A\ell}\tau_{\ell k}F^{-1}_{kB}\,,$$

$$\boldsymbol{T} = \frac{1}{2}\left(\boldsymbol{R}^T\cdot\boldsymbol{P} + \boldsymbol{P}^T\cdot\boldsymbol{R}\right), \qquad T_{AB} = \frac{1}{2}\left(R_{Ak}P_{kB} + P_{Ak}R_{kB}\right).$$

Solution: A comparison of equations (5.27) and (5.33) proves the rightfulness of the first relation. For the second equality the transformation based on (5.32) is the proof:

$$\boldsymbol{T} = \frac{1}{2}\left(\boldsymbol{U}\cdot\boldsymbol{S} + \boldsymbol{S}\cdot\boldsymbol{U}\right) = \frac{1}{2}\left(\boldsymbol{U}\cdot\boldsymbol{S} + (\boldsymbol{U}\cdot\boldsymbol{S})^T\right) = \underset{\boldsymbol{S}=\boldsymbol{F}^{-1}\cdot\boldsymbol{P}}{\uparrow} =$$
$$= \frac{1}{2}\left(\boldsymbol{U}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{P} + (\boldsymbol{U}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{P})^T\right) = \underset{\boldsymbol{F}^{-1}=\boldsymbol{U}^{-1}\cdot\boldsymbol{R}^T}{\uparrow} =$$
$$= \frac{1}{2}\left(\boldsymbol{U}\cdot\boldsymbol{U}^{-1}\cdot\boldsymbol{R}^T\cdot\boldsymbol{P} + (\boldsymbol{U}\cdot\boldsymbol{U}^{-1}\cdot\boldsymbol{R}^T\cdot\boldsymbol{P})^T\right) =$$
$$= \frac{1}{2}\left(\boldsymbol{R}^T\cdot\boldsymbol{P} + \boldsymbol{P}^T\cdot\boldsymbol{R}\right).$$

B.6. Problems in Chapter 6

Problem 6.1. Given the equilibrium condition of a body:

$$x_1 = (1+\alpha)X_1 + \beta X_2\,, \qquad x_2 = \beta X_1 + (1+\alpha)X_2\,, \quad x_3 = X_3 \tag{B.6.12}$$

where α and β are constants. Prove that

$$\rho = \frac{\rho^\circ}{(1+\alpha)^2 - \beta^2}\,. \tag{B.6.13}$$

Are there any restrictions on the constants α and β? If yes give them.
Solution: As is well known

$$\rho J = \rho^\circ$$

where

$$J = |F_{kA}| = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{vmatrix} = \begin{vmatrix} 1+\alpha & \beta & 0 \\ \beta & 1+\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1+\alpha)^2 - \beta^2$$

Hence

$$\rho = \frac{\rho^\circ}{(1+\alpha)^2 - \beta^2}\,.$$

Since ρ° and ρ are positive quantities the inequality

$$1+\alpha \geq \beta$$

should be fulfilled.

Problem 6.2. Given the velocity field

$$v_1 = \alpha x_1 - \beta x_2, \quad v_2 = \beta x_1 + \alpha x_2, \quad v_3 = \gamma\sqrt{x_1^2 + x_2^2}$$

in which α, β and γ are constants. Find the density in the current configuration provided that ρ° is known. Under what condition can this motion be isochoric?

Solution: It can be checked with ease that

$$\underset{(3\times 3)}{\underline{\mathbf{d}}} = \begin{bmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ \gamma\frac{x_1}{\sqrt{x_1^2+x_2^2}} & \gamma\frac{x_2}{\sqrt{x_1^2+x_2^2}} & 0 \end{bmatrix}, \quad d_I = \mathbf{v}\cdot\nabla = d_{11} + d_{22} = 2\alpha$$

are the velocity gradient and its divergence. Then the continuity equation has the form

$$\frac{d\rho}{dt} + \rho\,(\mathbf{v}\cdot\nabla) = \frac{d\rho}{dt} + \rho\,(\mathbf{v}\cdot\nabla) = \frac{d\rho}{dt} + 2\rho\alpha = 0\,,$$

from where

$$\varrho = \rho^\circ e^{-2t\alpha}.$$

is the solution that satisfies the initial condition $\rho = \rho^\circ$. The motion is isochoric if $\rho = \rho^\circ$ i.e, if $\alpha = 0$.

Problem 6.3. Prove using indicial notations that the Cauchy stress tensor is a symmetric tensor.

Solution: One can readily rewrite equation (6.19) in an indicial form:

$$\int_{V'} \rho\, e_{k\ell r} x_\ell b_r \,\mathrm{d}V + \int_{A'} e_{k\ell r} x_\ell t_{rs} n_s \,\mathrm{d}V = \int_{V'} \rho\, e_{k\ell r} x_\ell a_r \,\mathrm{d}V\,,$$

where

$$\begin{aligned}\int_{A'} e_{k\ell r} x_\ell t_{rs} n_s \,\mathrm{d}V &= \int_{V'} e_{k\ell r}\left(x_\ell t_{rs}\right)_{,s} \,\mathrm{d}V = \\ &= \int_{V'} e_{k\ell r}\, \underbrace{x_{\ell,s}}_{\delta_{\ell,s}}\, t_{rs} \,\mathrm{d}V + \int_{V'} e_{k\ell r} x_\ell t_{rs,s} \,\mathrm{d}V = \\ &= \int_{V'} e_{k\ell r}\, t_{r\ell} \,\mathrm{d}V + \int_{V'} e_{k\ell r} x_\ell t_{rs,s} \,\mathrm{d}V\,.\end{aligned}$$

Consequently,

$$\int_{V'} \Big[e_{k\ell r} x_\ell \underbrace{\left(t_{rs,s} + \rho b_r - \rho\, a_r\right)}_{=0} \Big] \,\mathrm{d}V + \int_{V'} e_{k\ell r}\, t_{r\ell} \,\mathrm{d}V = \int_{V'} e_{k\ell r}\, t_{r\ell} \,\mathrm{d}V = 0\,.$$

Since the subregion $V^{'}$ is arbitrary we get

$$e_{k\ell r}\, t_{r\ell} = 0\,.$$

This equation is the well known symmetry condition for the Cauchy stress tensor.

Problem 6.4. Assume that deformations are small. Assume further that we know the stress tensor which is given by equation (6.102). The coordinates X_ℓ are measured in mm.

(a) In the absence of body forces the stress field should be self-equilibrated. Check if the stress field (6.102) is really self equilibrated.

(b) Determine the stress vector at the point $\mathbf{X} = 2\,\mathbf{i}_1 + 3\,\mathbf{i}_1 + 2\,\mathbf{i}_3$ [mm] on the plane $2X_1 + X_2 - X_3 = 5$. (We remark that the unit normal to a plane defined by the equation $a_\ell X_\ell = b$ — a_ℓ and b are constants — is given by the relation $\mathbf{n} = a_\ell\, \mathbf{i}_\ell / \sqrt{a_k a_k}$.)

(c) Determine the normal stress $\sigma^{(n)}$ and the shearing stress $\boldsymbol{\tau}^{(n)}$ at the point we have given on the plane.

Solution: The stress field

$$\underset{(3\times 3)}{\underline{\mathbf{T}}} = \underset{(3\times 3)}{\underline{\boldsymbol{\sigma}}} = \alpha \begin{bmatrix} 6X_1X_3^2 & 0 & -2X_3^3 \\ 0 & 1 & 2 \\ -2X_3^3 & 2 & 6X_1^2 \end{bmatrix} \qquad \alpha = 1\mathrm{N/mm}^5$$

satisfies the symmetry condition. For the divergence of the stress field we obtain

$$\begin{aligned}\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \sigma_{12}}{\partial X_2} + \frac{\partial \sigma_{13}}{\partial X_3} &= 6X_3^2 + 0 - 6X_3^2 = \rho b_1 = f_1 = 0\,, \\ \frac{\partial \sigma_{21}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} + \frac{\partial \sigma_{23}}{\partial X_3} &= 0 + 0 + 0 = \rho b_2 = f_2 = 0\,,\end{aligned}$$

$$\frac{\partial\sigma_{31}}{\partial X_1}+\frac{\partial\sigma_{32}}{\partial X_2}+\frac{\partial\sigma_{33}}{\partial X_3}=0+0+0=\rho b_3=f_3=0$$

which means that the stress field is equilibrated if there are no body forces. Since

$$\mathbf{X}=2\mathbf{i}_1+3\mathbf{i}_2+2\mathbf{i}_3 \text{ mm} \quad \text{and} \quad \mathbf{n}=\frac{1}{\sqrt{6}}(2\mathbf{i}_1+\mathbf{i}_2-\mathbf{i}_3)$$

we get

$$\underset{(3\times3)}{\underline{\boldsymbol{\sigma}}}=\begin{bmatrix}48&0&-16\\0&1&2\\-16&2&24\end{bmatrix}\ \text{N/mm}^2\,,$$

$$\underset{(3\times1)}{\underline{\mathbf{t}}}^{(n)}=\frac{1}{\sqrt{6}}\begin{bmatrix}48&0&-16\\0&1&2\\-16&2&24\end{bmatrix}\begin{bmatrix}2\\1\\-1\end{bmatrix}=\frac{1}{\sqrt{6}}\begin{bmatrix}112\\-1\\-54\end{bmatrix}\ \text{N/mm}^2\,,$$

$$\sigma^{(n)}=\underset{(1\times3)}{\underline{\mathbf{n}}}^T\underset{(3\times1)}{\underline{\mathbf{t}}}^{(n)}=\frac{1}{6}\begin{bmatrix}2&0&-1\end{bmatrix}\begin{bmatrix}112\\-1\\-54\end{bmatrix}=\frac{139}{3}=46\frac{1}{3}\ \text{N/mm}^2\,,$$

$$\underset{(3\times1)}{\underline{\boldsymbol{\tau}}}^{(n)}=\underset{(3\times1)}{\underline{\mathbf{t}}}^{(n)}-\sigma^{(n)}\underset{(3\times1)}{\underline{\mathbf{n}}}=\frac{1}{\sqrt{6}}\begin{bmatrix}112\\-1\\-54\end{bmatrix}-\frac{46.33}{\sqrt{6}}\begin{bmatrix}2\\1\\-1\end{bmatrix}=\begin{bmatrix}33.20\\-6.67\\-15.8\end{bmatrix}\ \text{N/mm}^2\,.$$

Problem 6.5. Assume that the stress tensor within the body is given by the following equation:

$$\underset{(3\times3)}{\underline{\mathbf{t}}}=a\begin{bmatrix}x_1^2x_2 & x_1(b^2-x_2^2) & 0\\ x_1(b^2-x_2^2) & \frac{1}{3}x_2(x_2^2-3b^2) & 0\\ 0&0&2bx_3^2\end{bmatrix}$$

in which a and b are constants. Find the body forces if the stress tensor is equilibrated.

Solution: It is obvious that

$$f_1=\rho b_1=-\frac{\partial t_{11}}{\partial x_1}-\frac{\partial t_{12}}{\partial x_2}-\frac{\partial t_{13}}{\partial x_3}=-2ax_1x_2+2ax_1x_2+0=0\,,$$
$$f_2=\rho b_2=-\frac{\partial t_{21}}{\partial x_1}-\frac{\partial t_{22}}{\partial x_2}-\frac{\partial t_{23}}{\partial x_3}=-ab^2+ax_2^2-ax_2^2+ab^2+0=0\,,$$
$$f_3=\rho b_3=-\frac{\partial t_{31}}{\partial x_1}-\frac{\partial t_{32}}{\partial x_2}-\frac{\partial t_{33}}{\partial x_3}=0+0-4abx_3=-4abx_3\,.$$

Problem 6.6. Assume that

$$\underset{(3\times3)}{\underline{\mathbf{t}}}=\mu\alpha\begin{bmatrix}0&0&\frac{\partial\phi}{\partial x_1}-x_2\\0&0&\frac{\partial\phi}{\partial x_2}+x_1\\ \frac{\partial\phi}{\partial x_1}-x_2&\frac{\partial\phi}{\partial x_2}+x_1&0\end{bmatrix}.$$

is the stress field, where μ and α are non zero constants while $\phi(x_1,x_2)$ is a harmonic function, i.e., $\Delta\phi=0$. Does this stress tensor satisfy the equilibrium equations in the absence of body forces?

Solution: The solution procedures is the same as in the previous problem:

$$\rho b_1=0+0+\frac{\partial\left(\frac{\partial\phi}{\partial x_1}-x_2\right)}{\partial x_3}=0,$$
$$\rho b_2=0+0+\frac{\partial\left(\frac{\partial\phi}{\partial x_2}+x_1\right)}{\partial x_3}=0,$$

$$\rho b_3 = \frac{\partial\left(\frac{\partial\phi}{\partial x_1} - x_2\right)}{\partial x_1} + \frac{\partial\left(\frac{\partial\phi}{\partial x_2} + x_1\right)}{\partial x_1} + 0 = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} = \Delta\phi = 0.$$

This stress field is self equilibrated.

Problem 6.7. Given the function $f(\mathbf{x},t)$. Prove that

$$\int_V f(\mathbf{x},t)\mathbf{t}(\mathbf{x},t)\cdot\mathbf{n}\,\mathrm{d}V = \int_A \left[\mathbf{t}(\mathbf{x},t)\cdot(\nabla f(\mathbf{x},t)) + \rho f(\mathbf{x},t)(\mathbf{b}-\dot{\mathbf{v}})\right]\,\mathrm{d}A.$$

Solution: If we apply the divergence theorem we have

$$\begin{aligned}\int_V f(\mathbf{x},t)\mathbf{t}(\mathbf{x},t)\cdot\mathbf{n}\,\mathrm{d}V &= \int_A (f(\mathbf{x},t)\mathbf{t}(\mathbf{x},t))\cdot(\nabla))\,\mathrm{d}A = \\ &= \int_A \left[\left(\overset{\downarrow}{f}(\mathbf{x},t)\,\mathbf{t}(\mathbf{x},t)\right)\cdot\nabla) + \left(f(\mathbf{x},t)\,\overset{\downarrow}{\mathbf{t}}(\mathbf{x},t)\right)\cdot\nabla\right]\,\mathrm{d}A = \\ &= \int_A \left[\mathbf{t}(\mathbf{x},t)\cdot(\nabla f(\mathbf{x},t)) + \rho f(\mathbf{x},t)(\mathbf{b}-\dot{\mathbf{v}})\right]\,\mathrm{d}A.\end{aligned}$$

Problem 6.8. Given the equilibrium configuration of a body and the elements of the Cauchy stress tensor:

$$x_1 = 16X_1\,, \qquad x_2 = -\frac{1}{2}X_2\,, \quad x_3 = -\frac{1}{4}X_3$$
$$t_{11} = 100\ \text{MPa} \qquad t_{k\ell} = 0\ \text{ if }\ {}_{k\ell} \neq {}_{11}\,.$$

Determine (a) the first and second Piola-Kirchoff stress tensors then (b) the stress vector $\mathbf{t}^{(n)}$ and the pseudo stress vector $\mathbf{t}^{\circ(n)}$ on a plane with normal $\mathbf{n}^\circ$ before deformation.

Solution: It is obvious that

$$[F_{kA}] = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3}\end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.25\end{bmatrix} = [F_{Ak}]\,,$$

$$\left[F^{-1}_{B\ell}\right] = \begin{bmatrix} 0.0625 & 0 & 0 \\ 0 & -2.0 & 0 \\ 0 & 0 & -4.0\end{bmatrix} = \left[F^{-1}_{\ell B}\right],$$

$$J = |F_{kA}| = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3}\end{vmatrix} = \begin{vmatrix} 16 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.25\end{vmatrix} = 2.0$$

and

$$[t_{k\ell}] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}.$$

Consequently,

$$\begin{aligned}&[S_{AB}] = J\left[F^{-1}_{Ak}\right][t_{k\ell}]\left[F^{-1}_{\ell B}\right] = \\ &= 2.0\times\begin{bmatrix} 0.0625 & 0 & 0 \\ 0 & -2.0 & 0 \\ 0 & 0 & -4.0\end{bmatrix}\begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} 0.0625 & 0 & 0 \\ 0 & -2.0 & 0 \\ 0 & 0 & -4.0\end{bmatrix} = \\ &\qquad = \begin{bmatrix} 0.78125 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\end{aligned}$$

and

$$\mathbf{t}^{\circ(n)} = 0.781\,25 n_1^\circ \mathbf{i}_1$$

is the pseudo stress vector.

Problem 6.9. The mass center $\mathbf{x}_c$ of the body $\mathcal{B}$ is defined as

$$\mathbf{x}_c = \frac{1}{m}\int_V \rho \mathbf{x}\,\mathrm{d}V.$$

Prove that

$$m\frac{D^2\mathbf{x}_c}{Dt^2} = \int_V \rho\mathbf{b}\,\mathrm{d}V + \int_A \boldsymbol{t}\cdot\mathbf{n}\,\mathrm{d}A\,.$$

Solution: It is obvious that

$$\frac{D^2\mathbf{x}_c}{Dt^2} = \frac{1}{m}\int_V \rho\mathbf{a}\,\mathrm{d}V = \frac{1}{m}\int_V (\boldsymbol{t}\cdot\nabla + \rho\mathbf{b})\,\mathrm{d}V$$

from where it follows the equation to be proved at once if we apply the divergence theorem.

Problem 6.10. Assume that the stress field is self equilibrated. Prove that the average value of the Cauchy stress tensor

$$\bar{t}_{\ell k} = \frac{1}{V}\int_A t_{\ell k}\,\mathrm{d}V$$

can be calculated as

$$\bar{t}_{\ell k} = \frac{1}{2V}\int_V \rho\left(x_k b_\ell + x_\ell b_k\right)\,\mathrm{d}V + \frac{1}{2V}\int_A \left(x_k t_\ell^{(n)} + t_k^{(n)} x_\ell\right)\mathrm{d}A\,.$$

Solution: Utilizing equilibrium equation (6.25a) the following manipulation proves the statement of the problem:

$$\begin{aligned}
&\frac{1}{V}\int_V t_{\ell k}\,\mathrm{d}V = \frac{1}{2V}\int_V \left(t_{\ell k} + t_{k\ell}\right)\,\mathrm{d}V = \\
&= \frac{1}{2V}\int_V \Big[t_{\ell k} + \underbrace{x_k\left(t_{\ell r,r} + \rho b_\ell\right)}_{=0} + t_{k\ell} + \underbrace{x_\ell\left(t_{kr,r} + \rho b_k\right)}_{=0}\Big]\,\mathrm{d}V = \\
&= \frac{1}{2V}\int_V \Big[t_{\ell k} - \delta_{kr}t_{\ell r} + \left(x_k t_{\ell r}\right)_{,r} + \rho x_k b_\ell + t_{k\ell} - \delta_{\ell r}t_{kr} + \left(x_\ell t_{kr}\right)_{,r} + \rho x_\ell b_k\Big]\,\mathrm{d}V = \\
&\qquad = \frac{1}{2V}\int_V \rho\left(x_k b_\ell + x_\ell b_k\right)\,\mathrm{d}V + \frac{1}{2V}\int_A \left(x_k t_{\ell r} n_r + x_\ell t_{kr} n_r\right)\mathrm{d}A = \\
&\qquad\qquad = \frac{1}{2V}\int_V \rho\left(x_k b_\ell + x_\ell b_k\right)\,\mathrm{d}V + \frac{1}{2V}\int_A \left(x_k t_\ell^{(n)} + t_k^{(n)} x_\ell\right)\mathrm{d}A\,.
\end{aligned}$$

Problem 6.11. Prove the validity of transformation (6.88) using indicial notation.

Solution: Making use of (6.88) we may write:

$$\begin{aligned}
\boldsymbol{S}\cdot\cdot\left(\boldsymbol{E}\right)^{\boldsymbol{\cdot}} &= S_{AB}\left(E_{AB}\right)^{\boldsymbol{\cdot}} = F_{A\ell}^{-1}P_{\ell B}\,\frac{1}{2}\left(\left(F_{Am}\right)^{\boldsymbol{\cdot}}F_{mB} + F_{Am}\left(F_{mB}\right)^{\boldsymbol{\cdot}}\right) = \\
&= F_{A\ell}^{-1}P_{\ell B}\,F_{Am}\left(F_{mB}\right)^{\boldsymbol{\cdot}} = \delta_{\ell m}P_{\ell B}\left(F_{mB}\right)^{\boldsymbol{\cdot}} = P_{\ell B}\left(F_{\ell B}\right)^{\boldsymbol{\cdot}} = \boldsymbol{P}\cdot\cdot\left(\boldsymbol{F}\right)^{\boldsymbol{\cdot}}\,.
\end{aligned}$$

B.7. Problems in Chapter 7

Problem 7.1. Prove the principle of complementary virtual power.

Solution: The proof is simple if one repeats the line of thought for the proof of the principle of complementary virtual work being presented in Subsection 7.4.3.

B.8. Problems in Chapter 8

Problem 8.1. Assume that the function ϕ is an isotropic function of the symmetric tensor $\boldsymbol{E}$. Prove that the derivative $\partial\phi/\partial\boldsymbol{E}$ is coaxial with $\boldsymbol{E}$.

Solution: If ϕ is an isotropic function of $\boldsymbol{E}$ then it should be a function of the scalar invariants E_I, E_{II} and E_{III} of $\boldsymbol{E}$:

$$\phi = \phi(E_I, E_{II}, E_{III}).$$

By repeating the line of thought that leads from equation (8.116) to equation (8.119) we may conclude that

$$\frac{\partial\phi}{\partial E_{ab}} = a_\circ^\circ \delta_{ab} + a_1^\circ E_{ab} + a_2^\circ E_{aq} E_{qb}\,,$$

where the coefficients $a_\circ^\circ$, a_1° and a_2° are given by equations (8.120). The above tensor polynomial is obviously coaxial with the tensor $\boldsymbol{E}$.

Problem 8.3. Prove that the number of independent components in $\mathcal{C}_{mnk\ell}$ is 21.

Solution: According to equation (8.86)

$$\mathcal{C}_{mnk\ell} = \rho^\circ \frac{\partial^2 f(\Theta_\circ, \mathbf{0})}{\partial\varepsilon_{mn}\partial\varepsilon_{k\ell}}\,.$$

Since $\varepsilon_{mn} = \varepsilon_{nm}$ and $\varepsilon_{k\ell} = \varepsilon_{\ell k}$ it follows that

$$\mathcal{C}_{mnk\ell} = \mathcal{C}_{nmk\ell} \qquad \text{and} \qquad \mathcal{C}_{mnk\ell} = \mathcal{C}_{mn\ell k}.$$

The number of independent components is, therefore, $6 \times 6 = 36$. Since the derivation order has no effect on the result it also follows that there is a symmetry in respect of the index pairs mn and kl:

$$\mathcal{C}_{mnk\ell} = \mathcal{C}_{k\ell nm}.$$

This condition results in that the components with the indices

1122 ⇔ 2211	1112 ⇔ 1211	2212 ⇔ 1222	3312 ⇔ 1233
1133 ⇔ 3311	1123 ⇔ 2311	2223 ⇔ 2322	3323 ⇔ 2333
2233 ⇔ 3322	1131 ⇔ 3111	2231 ⇔ 3122	3331 ⇔ 3133
	1213 ⇔ 1312	1223 ⇔ 2312	1323 ⇔ 2312

are the same. This means that the number of independent tensor components is further reduced by fifteen. The final result for the number of independent components is, therefore, 36-25=21.

Problem 8.5. Verify that equation (8.146) is correct.

Solution: Substituting b for W in equation (1.129) yields:

$$b_{km} b_{m\ell} = b_{III} b_{k\ell}^{-1} + b_I b_{k\ell} - b_{II} \delta_{k\ell}$$

Hence

$$t_{k\ell} = \frac{2}{J}\left[b_{III}\frac{\partial\psi}{\partial b_{III}}\delta_{k\ell} + \left(\frac{\partial\psi}{\partial b_I} + b_I\frac{\partial\psi}{\partial b_{II}}\right)b_{k\ell} - \left(\frac{\partial\psi}{\partial b_{II}}\right)\underbrace{\left(b_{III}b_{k\ell}^{-1} + b_I b_{k\ell} - b_{II}\delta_{k\ell}\right)}_{b_{km}b_{m\ell}}\right] =$$

$$= \frac{2}{J}\left[\left(b_{II}\frac{\partial\psi}{\partial b_{II}} + b_{III}\frac{\partial\psi}{\partial b_{III}}\right)\delta_{k\ell} + \frac{\partial\psi}{\partial b_I}b_{k\ell} - b_{III}\frac{\partial\psi}{\partial b_{II}}b_{k\ell}^{-1}\right].$$

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