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Rough Sets and Their Lattice-theoretic Analysis

Thesis booklet

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1 Overview of Research Goals

In my doctoral dissertation, I discuss the lattice-theoretical properties and application possibilities of generalized versions of the rough set model. From a scientific viewpoint, the theory of rough sets contains a lot of research possibilities. Even though the basic rough set model has been discussed extensively over the previous decades, more general models appeared opening up new research directions. The analysis of the order structures of such models provided a research gap I could contribute to with my work. For example, there were not many discussions of the lattice structures of multigranular rough sets defined using more than two equivalence relations. Additionally, the conditions for the combined fuzzy-rough models forming a lattice or a complete lattice, and the algebraic-logic structures that could be defined over them have not been discussed in detail previously.

Furthermore, the practical applicability of rough sets also provides a wide range of possibilities for research. In the past, rough sets have successfully been applied for data processing, image processing, data mining and preprocessing of input data for neural networks. The more general rough set models discussed in my dissertation not only open the gates for novel applications, but also for improving existing applications. Moreover, it is important to know the properties of these models for both theoretical discussion and practical applications. These properties also determine what logical approach can be used.

Therefore, the aim of my research was to examine the more general models of rough sets from a lattice-theoretical viewpoint, and to determine their application possibilities. One of these generalized models is the concept of multigranular rough sets. I examined two versions of this model, optimistic and pessimistic. With the analysis of their order structures, my goal was to determine when the multigranular rough sets form a lattice, and what the conditions are for this lattice to have specific useful properties. Additionally, since the optimistic case is only a poset in general, I discuss its Dedekind–MacNeille completion as well.

I discussed three ways of another type of generalization, namely the combination with fuzzy sets. When the reference set is crisp and the approximation space is fuzzy, I aimed to determine the conditions for this model to have the same structural properties as regular rough sets. For the fuzzy sets and fuzzy relations in the model, I also examined the relationship between their cores and supports, and the relationship between their exact (definable) sets.

I also analyzed the case when the reference set is fuzzy (more precisely, L-fuzzy) and the approximation space is crisp. On one hand, I aimed to determine what properties are inherited from the lattice L to the lattice of rough L-fuzzy sets, and when they form three-valued Łukasiewicz-algebras. In my research plans, I also aimed to determine the conditions for a pair of fuzzy sets to form a rough L-fuzzy set. This resulted in the statement and the proof of a representation theorem.

The previous two combined models paved the way for the research of the more general version, where both the approximation space and the reference set are fuzzy. For this model, I aimed to present a representation theorem for conditions that result in a pair of fuzzy sets forming a fuzzy rough set. Furthermore, I revealed the set of conditions for the fuzzy sets to be a lattice or a complete lattice.

Finally, I aimed to develop a novel application for fuzzy rough sets. For this, I proposed an image compression method greatly improving an existing method by incorporating the more general fuzzy rough sets. To compare my results to the original method, I used the PSNR value for the images reconstructed from the compressed data. My proposed method consistently outperformed the previous method.

2 Description of Analysis

To examine the more general versions of rough set theory, I used the mathematical tools provided by the theory of partially ordered sets, lattice theory and other algebraic methods. In the next three subsections, I will introduce these. In the last subsection, I will discuss the preliminary theory and the necessary tools for the proposed image compression method and its evaluation.

2.1 Lattice Theoretical Background

The basic concepts of lattice theory and algebra are summarized well in many books. The most relevant for my research topic will be demonstrated based on [20], [16] and a [9].

Let U be a universe, and let $\varrho \subseteq U \times U$ be a *homogeneous binary relation* (in what follows, simply *relation*). There are two ways to denote that elements $a, b \in U$ are in relation: $(a, b) \in \varrho$ or using infix notation $a\varrho b$. We say that the relation ϱ is a *partial order* if it is reflexive, transitive and antisymmetric. A relation is called an *equivalence* if it is reflexive, transitive and symmetric. Omitting one of these properties yields some new concepts. Reflexive and transitive relations are called *quasi-orders*. Reflexive and symmetric relations are called *tolerance* relations. Now, let P be a set and let $\leq$ be a partial order on it. Then the structure $(P, \leq)$ is called a *partially ordered set* or *poset*.

Let $(L, \leq)$ be a partially ordered set. If any two elements $a, b \in L$ have an *infimum*, $\inf\{a, b\}$ (greatest lower bound) and a *supremum*, $\sup\{a, b\}$ (least upper bound), then we call $(L, \leq)$ a *lattice*. If any $A \subseteq L$ has an infimum, $\inf A$ and a supremum, $\sup A$, then $(L, \leq)$ is a *complete lattice*. The infimum and supremum of any two elements $a, b \in L$ is also denoted by $a \wedge b$ and $a \vee b$, respectively. Similarly the infimum and supremum of $A \subseteq L$ are denoted by $\bigwedge A$ and $\bigvee A$, respectively. We say that the lattice $(L, \leq)$ is distributive if the following two (equivalent) equalities hold:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

A complete lattice L is *completely distributive* if for any doubly indexed $x_{i,j} \in L$ elements ($i \in I, j \in J$ and $I, J \neq \emptyset$), the following hold:

$$\bigwedge_{i \in I} \left(\bigvee_{j \in J} x_{i,j} \right) = \bigvee_{f\colon I \to J} \left(\bigwedge_{i \in I} x_{i,f(i)} \right). \tag{2.1}$$

A lattice L with least element 0 is called *pseudocomplemented* if for each $x \in L$ there exists a unique element $x^* \in L$ such that $x \wedge z = 0$ holds if and only if $z \leq x^*$. x^* is called the *pseudocomplement* of x in L. A *Stone lattice* is a distributive pseudocomplemented lattice satisfying $x^* \vee x^{**} = 1$ for all

$x \in L$. $x^+ \in L$ is called the *dual pseudocomplement* of $x \in L$ if $x \vee z = 1$ is equivalent to $z \geq x^+$. If every element in L has a pseudocomplement and a dual pseudocomplement, then L is called a *double pseudocomplemented lattice*. A *double Stone lattice* is a Stone lattice where each element $x \in L$ has a dual pseudocomplement $x^+ \in L$ and satisfies $x^+ \wedge x^{++} = 0$. A double pseudocomplemented lattice is *regular* if $x^* = y^*$ and $x^+ = y^+$ imply $x = y$ for all $x, y \in L$.

2.2 Theory of Rough Sets

The notion of rough sets was introduced by Zdzisław Pawlak [66]. The main idea behind this notion is to handle the incomplete information about some objects. Due to information incompleteness, some of the objects may be indistinguishable from each other. In the original rough set model, this is represented by an indiscernibility relation, which is an equivalence. Over the years, the rough set model has been generalized and extended in many ways.

An overview of the first decade of development of rough sets can be found in [69], and some further research directions and applications are mentioned in [70]. Originally, Pawlak assumed that this relation is an equivalence, but later several other types of relations were also examined [75, 92, 36, 40, 42, 38, 48, 49, 109]. One generalization approach is to use another type of binary relation instead of an equivalence, for instance, quasi-orders or tolerances [109, 38, 40].

The concept of rough sets extends the notion of the traditional (*crisp*) set by taking into account that the information we have is usually not complete, and uncertain in many cases.

Let A be a subset of universe U and let $E \in U \times U$ be an equivalence. Then (U, E) is called an *approximation space*, in which the *lower and upper approximations* of reference set $A \subseteq U$ are defined as

$$A_E = \{x \in U \mid [x]_E \subseteq A\}, A^E = \{x \in U \mid [x]_E \cap A \neq \emptyset\}, \tag{2.2}$$

respectively, where $[x]_E$ is the equivalence class of element $x \in U$. We can define an equivalence relation $\equiv$ on the subsets of U:

$$A \equiv B \Leftrightarrow A_E = B_E \text{ and } A^E = B^E,$$

where $A, B \subseteq U$. Thus, we can identify the *rough set* of $A \subseteq U$ as the pair (A_E, A^E).

A set that is a union of the classes of equivalence E are called *E-definable*, or *exact* sets. Note that if A is an E-definable set, then $A_E = A^E = A$.

Instead of an equivalence, an arbitrary $R \in U \times U$ relation can be used to define the approximation operators::

$$A_R = \{x \in U \mid R(x) \subseteq A\}, A^R = \{x \in U \mid R(x) \cap A \neq \emptyset\}, \tag{2.3}$$

where $R(x) = \{y \in U \mid (x, y) \in R\}$ for $x \in U$. In case of an equivalence, we have $R(x) = [x]_R$. The set of all rough sets in approximation space (U, R) is denoted by $\mathcal{RS}(U, R)$, its elements being ordered componentwise:

$$(A_R, A^R) \leq (B_R, B^R) \Leftrightarrow A_R \subseteq B_R \text{ and } A^R \subseteq B^R. \tag{2.4}$$

In general $(\mathcal{RS}(U, R), \leq)$ is a bounded poset with least element $(\emptyset, \emptyset)$, and greatest element (U, U).

The *set of all lower approximations* is denoted by $\mathcal{P}(U)_R$, the *set of all upper approximations* is denoted by $\mathcal{P}(U)^R$. It is known that $\mathcal{P}(U)_R$ and $\mathcal{P}(U)^R$ form dually isomorphic lattices. In case R is a quasi-order, $\mathcal{P}(U)_R$, $\mathcal{P}(U)^R$ and $\mathcal{RS}(U, R)$ form completely distributive lattices.

Theorem 2.1 ([76, 14])**.** *Let $E \in U \times U$ be an equivalence. Then $(\mathcal{RS}(U, E), \vee, \wedge, \sim, (\emptyset, \emptyset), (U, U))$ is a regular double Stone lattice.*

2.3 Fuzzy Sets and Fuzzy Relations

The theory of fuzzy sets was introduced by Lotfi Zadeh [112]. A fuzzy set A is defined by a membership function $\mu_A : U \longrightarrow [0, 1]$. The membership degree 0 means that the element is a member of the set A with zero measure, and the membership degree 1 means that the element is in the set with full measure. We say that f has a *finite range*, whenever the (crisp) set $\{\mu_A(x) \mid x \in U\}$ is finite. The collection of all fuzzy sets on U is denoted by $\mathcal{F}(U)$. A *fuzzy relation* R on universe U is a fuzzy set defined on $U \times U$, meaning that its membership function is $\mu_R : U \times U \to [0, 1]$. We say that fuzzy relation R has a finite range if the (crisp) set $\{\mu_R(x, y) \mid x, y \in U\}$ is finite.

In what follows, to simplify notations I may only use the membership function to refer to a fuzzy set or a fuzzy relation, e.g. the $f : U \to [0, 1]$ fuzzy set, the $\theta : U \times U \to [0, 1]$ fuzzy relation.

A *fuzzy similarity relation* (alternatively, a *fuzzy equivalence relation*) is a reflexive, symmetric and transitive fuzzy relation. As we are considering fuzzy relations, *reflexive property* means that $\mu_R(x, x) = 1$ for every $x \in U$ and *symmetry* means that $\mu_R(x, y) = \mu_R(y, x)$ for every $x, y \in U$. Initially, a fuzzy relation R was called *transitive* if $\min(\mu_R(x, y), \mu_R(y, z)) \leq \mu_R(x, z)$, for all $x, y, z \in U$ [113]. Later this notion was generalized by using the notion of a t-norm [99]. Defining $\sim x := 1 - x$ in lattice $([0, 1], \leq)$, we get a completely distributive Kleene algebra.

A triangular norm $\mathcal{T}$ (*t-norm* for short) is an increasing commutative and associative mapping $\mathcal{T}\colon [0, 1]^2 \longrightarrow [0, 1]$ satisfying $\mathcal{T}(1, x) = \mathcal{T}(x, 1) = x$, for all $x \in [0, 1]$. The t-norm $\mathcal{T}$ is called *positive* [5] if $\mathcal{T}(x, y) > 0$, whenever $x, y > 0$. We say that a fuzzy relation $\mu_R\colon U^2 \longrightarrow [0, 1]$ is $\mathcal{T}$*-transitive*, if

$$\mathcal{T}(\mu_R(x, y), \mu_R(y, z)) \leq \mu_R(x, z), \text{ for all } x, y, z \in U.$$

A reflexive, symmetric and $\mathcal{T}$-transitive fuzzy relation R is called a *fuzzy $\mathcal{T}$-equivalence,* or a *fuzzy $\mathcal{T}$-similarity relation.*

It is well-known that $\mathcal{T}(x, y) = \min(x, y)$, $x, y \in [0, 1]$ is a positive t-norm corresponding to the previous notion of transitivity. Thus, when $\mathcal{T}(x, y) = \min(x, y)$, then we still call it simply a *fuzzy similarity (equivalence) relation*. Every $\mathcal{T}$ t-norm satisfies $\mathcal{T}(x, 0) = \mathcal{T}(0, x) = 0$.

The t-norm $\odot$ is called *(left) continuous* (see [25]), if it is (left) continuous as a function $\odot\colon [0, 1]^2 \to [0, 1]$ in the usual interval topology on $[0, 1]^2$. Infix notation can also be used when applying t-norms, e.g. instead of $\odot(x, y)$ we can write $x \odot y$.

A *triangular conorm* $\oplus$ (shortly *t-conorm*) [84] is a commutative, associative and monotone increasing binary operation defined on $[0, 1]$ satisfying $0 \oplus x = x \oplus 0 = x$ for all $x \in [0, 1]$. The t-conorm $\oplus$ is *(left) continuous* [25], if it is (left) continuous as a function $\oplus\colon [0, 1]^2 \to [0, 1]$ in the usual topology.

2.4 Tests for Image Compression

To test my proposed image compression algorithm, I implemented my improved procedure, alongside the original algorithm of Petrosino and Ferone [73]. The programming was done in C#, the tests were using a subset of the *2017 Unlabeled images* dataset from *Microsoft COCO: Common Objects in Context* [50].

To measure the quality of the reconstructed image, I calculated the *peak signal-to-noise ratio* (*PSNR*) values – a metric commonly used in signal processing [28]. Let the width of the image be w, the height of the image be h, and let the value of color channel $c \in \{R, G, B\}$ on the original image for the pixel on row y and column x be $I_c(x, y)$. For the compressed image, let the same pixel color value be $C_c(x, y)$. In practice, the *mean squared error* is generally used to quantify the difference between images:

$$MSE = \frac{1}{3 \cdot w \cdot h} \sum_{c \in \{R,G,B\}} \sum_{x=0}^{w} \sum_{y=0}^{h} \left(I_c(x, y) - C_c(x, y)\right)^2. \tag{2.5}$$

Then the peak signal-to-noise ratio (in decibels) is given by the following formula:

$$PSNR = 10 \cdot \log_{10}\left(\frac{MAX^2}{MSE}\right), \tag{2.6}$$

where MAX denotes the number of possible values the given pixel's color channel can take (in other words, the maximal value of the binary representation of the color channel). Since the processed images use 8-bit channels, $MAX = 2^8 - 1 = 255$. The closer the reconstructed image is to the original, the higher the PSNR value.

3 Scientific Results

3.1 Multigranular Rough Sets

3.1.1 Theoretical Overview

The theory of rough sets can be viewed as representing the opinion of a single expert about what elements are indistinguishable from each other. In this sense, the multigranular rough set model could mean that we are considering the opinion of multiple experts. Thus, the indiscernibility of objects will also be expressed by multiple equivalence relations. The optimistic and pessimistic multigranular models are two common ways to model this.

Let $P_1, P_2, \ldots, P_n$ be equivalence relations on universe U. Then the so-called *optimistic lower approximation* of a set $X \subseteq U$ [78] is defined as

$$X_{\sum_{i=1}^{n} P_i} = \{x \in U \mid P_1(x) \subseteq X \text{ or } P_2(x) \subseteq X \text{ or ... or } P_n(x) \subseteq X\}\,. \tag{3.1}$$

For brevity and readability, in the rest of the text, I use $\sum P_i$ to mean $\sum_{i=1}^{n} P_i$ in what follows. The *optimistic upper approximation* of X is defined as:

$$X^{\sum P_i} = \{x \in U \mid P_1(x) \cap X \neq \emptyset \text{ and ... and } P_n(x) \cap X \neq \emptyset\}\,. \tag{3.2}$$

Let $\mathcal{P}(U)^{\sum P_i}$ and $\mathcal{P}(U)_{\sum P_i}$ denote all optimistic upper approximations and all optimistic lower approximations defined on U, respectively.

The pessimistic lower and upper approximations are given by applying formula 2.3 using the tolerance relation $\bigcup_{i=1}^{n} P_i$.

3.1.2 Main Results

In my research, I defined the following conditions for equivalences $P_1, P_2, \ldots, P_n \subseteq U \times U$.

($\star$) For all $x \in U$, there exists $k_x \in \{1, 2, ..., n\}$ such that $P_{k_x}(x) \subseteq P_i(x)$ for all $i \in \{1, 2, ..., n\}$.

$(\star)^d$ For all $x \in U$, there exists $k_x \in \{1, ..., n\}$ such that $P_i(x) \subseteq P_{k_x}(x)$ for all $i \in \{1, ..., n\}$.

(Si) $\left|\left(\bigcap_{i=1}^{n} P_i\right)(x)\right| = 1 \Rightarrow$ there exists $k \in \{1, 2, ..., n\}$ such that $|P_k(x)| = 1$.

Using these I proved the following statements.

Proposition 3.1. *Let $P_1, ..., P_n$ be equivalence relations on U satisfying ($\star$). Then RS $(\sum P_i)$ is a completely distributive regular double Stone lattice.*

Proposition 3.2. *If $P_1, P_2, ..., P_n$ are equivalence relations on universe U satisfying condition $(\star)^d$, then $\bigcup P_i$ is an equivalence relation.*

Corollary 3.3. *Let $P_1, P_2, ..., P_n$ be equivalence relations on universe U that satisfy $(\star)^d$. Then $RS\left(\bigcup P_i\right)$ is a completely distributive regular double Stone lattice.*

Theorem 3.4. *Let $P_1, P_2, ...P_n$ be equivalence relations on universe U satisfying condition (Si). Then $RS\left(\sum P_i\right)$ is a complete lattice.*

Additionally, define the *increasing representation of $\sum P_i$-rough sets* as

$$IRS(\sum P_i) =$$

$$\left\{(A, B) \in \mathcal{P}(U)_{\sum P_i} \times \mathcal{P}(U)^{\sum P_i} \mid A \subseteq B \text{ és } (B \setminus A) \cap \left(\bigcup_{i=1}^{n} S_i\right) = \emptyset\right\}.$$

Regarding this set, I proved the following theorem.

Theorem 3.5. *$IRS\left(\sum P_i\right)$ is the Dedekind-MacNeille completion of $RS\left(\sum P_i\right)$.*

Additionally, using some lemmas and previous propositions, I stated and proved the following.

Proposition 3.6. *The following statements are equivalent:*

(a) $IRS\left(\sum P_i\right)$ is distributive;

(b) $\mathcal{P}(U)_{\sum P_i}$ and $\mathcal{P}(U)^{\sum P_i}$ are distributive;

(c) $P_1, P_2, ..., P_n$ satisfy condition $(\star)$;

(d) $\mathcal{P}(U)_{\sum P_i}$ and $\mathcal{P}(U)^{\sum P_i}$ form Alexandroff topologies.

(e) $RS\left(\sum P_i\right)$ is a completely distributive regular double Stone lattice.

3.1.3 Novelty and Applicability

Previously, the order structures of multigranular rough sets were not examined in detail for the general case, i.e. for $n \geq 2$. Generalizing the results for $n = 2$ in [41] contributes greatly to this scientific area as it opens up the possibility to discuss even more general cases. Furthermore, the usefulness of examining the properties of multigranular rough sets is apparent not only the theoretical analysis, but in practical applicability as well. This model can be utilized e.g. in recommender systems.

3.1.4 I. Thesis

I investigated the order structures of optimistic and pessimistic multigranular rough sets. I determined under what conditions they form complete lattices and completely distributive lattices. I showed that the increasing representation of optimistic multigranular rough sets coincides with the Dedekind–MacNeille completion. I identified the conditions under which pessimistic and optimistic multigranular rough sets form a completely distributive regular double Stone lattice. (The classical Pawlakian rough sets also form such a lattice.)

My publications concerning this thesis: [S7, S6, S5]

3.2 Fuzzy rough sets with fuzzy reference sets

3.2.1 Theoretical Overview

Since the introduction of rough set theory, there have been many ways to combine the notion with the theory of fuzzy sets. During my research, I first examined the type of fuzzy rough sets where the approximation space is fuzzy, but the reference set is crisp.

Let (U, R) be a fuzzy approximation space, where U is the universe and R is a fuzzy similarity relation given by the membership function $\mu_R : U^2 \longrightarrow [0, 1]$. Furthermore, let $A \subseteq U$ be a crisp set. The lower approximation of A is given by the membership function

$$\mu_{[A]_R}(x) = \inf\{1 - \mu_R(x, y) \mid y \notin A\},$$

and the upper approximation of A is given by

$$\mu_{[A]^R}(x) = \sup\{\mu_R(x, y) \mid y \in A\}.$$

The *fuzzy rough set* corresponding to reference set A is defined as the pair $(\mu_{[A]_R}, \mu_{[A]^R})$. It is known that the set of all rough sets in approximation space (U, R) form a lattice with several interesting properties [76, 14, 23, 26].

The set $S_R := \{\mu_R(x, y) \mid x, y \in U\} \subseteq [0, 1]$ is called the *spectrum* of R. We say that a fuzzy relation R has a *dually well-ordered spectrum*, if any nonempty subset of S_R has a maximal element.

Let us define the following crisp equivalence relations for the fuzzy similarity R:

$$E := \{(x, y) \in U^2 \mid \mu_R(x, y) = 1\}, \quad S = \{(x, y) \in U^2 \mid \mu_R(x, y) > 0\}.$$

The former is called the *core* of R, the latter is called the support of R. (The notion is defined for fuzzy sets analogously.)

3.2.2 Main Results

I determined the relationship between the cores and supports of fuzzy approximations and the core and support of R, the fuzzy similarity relation of the approximation space. As a result, I stated the following lemma.

Lemma 3.7. *For any subset $A \subseteq U$ we have*

$$\text{(i) } A^E = \{x \in U \mid \mu_{[A]^R}(x) = 1\}, \quad \text{(ii) } A_E = \{x \in U \mid \mu_{[A]_R}(x) > 0\},$$

$$\text{(iii) } A^S = \{x \in U \mid \mu_{[A]^R}(x) > 0\}, \quad \text{(iv) } A_S = \{x \in U \mid \mu_{[A]_R}(x) = 1\}.$$

Let the set of fuzzy rough sets with crisp reference sets be denoted by $(\mathcal{RS}(U, R), \leq)$. Let the set of (traditional) rough sets for E (the core of R) be denoted by $(\mathrm{RS}(U, E), \leq)$. I made the connection between the two by stating and proving the following theorem.

Theorem 3.8. *Let R be a fuzzy $\mathcal{T}$-equivalence on U with a dually well-ordered spectrum. Then $(\mathcal{RS}(U, R), \leq)$ is a complete lattice isomorphic to $(RS(U, E), \leq)$.*

An immediate consequence is the following:

Corollary 3.9. *If R is a fuzzy equivalence on the set U with a dually well-ordered spectrum, then $(\mathcal{RS}(U, R), \leq)$ is a completely distributive regular double Stone lattice.*

The concept of exact sets can be extended for fuzzy rough sets as well. A fuzzy rough set defined by the fuzzy equivalence R is *exact* if for every $x \in U$, $\mu_{[A]_R}(x) = \mu_{[A]^R}(x)$ holds, where U is the universe of R. The following proposition describes the relationship between exact fuzzy rough sets and the support of the fuzzy equivalence relation.

Proposition 3.10. *Let A be a (crisp) subset of U. Then*

$$\mu_{[A]_R}(x) = \mu_{[A]^R}(x) \text{ for all } x \in U \Leftrightarrow A_S = A^S.$$

3.2.3 Novelty and Applicability

The lattice-theoretic discussion of fuzzy rough sets (whether with crisp or fuzzy reference sets) was a research gap, thus the results are novel, opening the way for further analysis. Determining when fuzzy sets with crisp reference sets form a completely distributive regular double Stone lattice is important, because then they behave like regular rough sets. This also means that the same logic structures can be defined on them. This research was also useful for the later investigation of fuzzy rough sets with fuzzy reference sets, i.e. the topic of Thesis IV.

3.2.4 II. Thesis

I showed that the lattice of fuzzy rough sets with crisp reference sets and defined over a fuzzy similarity relation with a dually well-ordered spectrum forms a complete lattice, which is isomorphic to the lattice of rough sets defined over the core of the fuzzy relation. I examined the relationship between the support and core of the lower and upper approximations of such fuzzy rough sets, and the approximations defined over the support and core of the fuzzy relation. I proved that the exact sets of the support of the fuzzy similarity relation—more precisely, their characteristic functions—coincide with the exact fuzzy rough sets (with a crisp reference set).

My publications concerning this thesis: [S8, S4]

3.3 Rough L-fuzzy sets

3.3.1 Theoretical Overview

Instead of the traditional $[0,1]$ interval, membership functions can take upon the values from a bounded L lattice. Let U be a non-empty universe, $f\colon U \to L$ be an L-fuzzy set and $E \subseteq U \times U$ be an equivalence on U. The *lower approximation* $\underline{f}\colon U \to L$ and *upper approximation* $\overline{f}\colon U \to L$ of f are defined as:

$$\underline{f}(x) := \bigwedge\{f(y) \mid y \in [x]_E\}, \quad x \in U, \tag{3.3}$$

$$\overline{f}(x) := \bigvee\{f(y) \mid y \in [x]_E\}, \quad x \in U, \tag{3.4}$$

The collection of L-fuzzy sets is denoted by $\mathcal{F}(U,L)$, the set of all rough L-fuzzy sets is denoted by $\mathcal{RF}(U,L)$. It is known that $\mathcal{F}(U,L)$ is a complete lattice.

The *singletons* of the (crisp) equivalence E are elements of the set $S := \{s \in U \mid [s]_E = \{s\}\}$.

We say that the map $\alpha\colon U \to L$ is *constant on the classes of the equivalence* E, (or shortly E*-constant*) whenever $(x,y) \in E$ implies $\alpha(x) = \alpha(y)$, for any $x, y \in U$.

The approximation space $(U_1 \times U_2, R)$ is called the *direct product* of the approximation spaces (U_1, R_1) and (U_2, R_2). For any relation $\varrho \subseteq U_1 \times U_2$ its *lower* and its *upper approximation* in $(U_1 \times U_2, R)$ are defined the usual way.

An element $x \neq 0$ of a complete lattice L is *completely join-irreducib*le if for any $T \subseteq L$, $x = \bigvee T$ implies $x \in T$. The set of completely join-irreducible elements of L is denoted by J. Furthermore, let $\triangle_L := \{(x,x) \mid x \in L\}$.

For any $x \in L$, let $J(x) = \{j \in J \mid j \leq x\}$ and $A(x) = \{a \in A \mid a \leq x\}$. A lattice L is *spacial* (*atomistic*) if for any $x \in L$, $x = \bigvee J(x)$ ($x = \bigvee A(x)$).

3.3.2 Main Results

Theorem 3.11. *Let L be a complete lattice, and $U \neq \emptyset$ a universe. A pair (f, g) of L-fuzzy sets $f, g\colon U \to L$ is a rough L-fuzzy set induced by an equivalence $E \subseteq U \times U$ if and only if the following conditions are satisfied:*

(a) f and g are constant on the classes of E;

(b) $f \leq g$;

(c) $f(s) = g(s)$, for all $s \in S$.

Theorem 3.12. *Let E be an equivalence on a universe $U \neq \emptyset$ and L a complete spacial distributive lattice. Then $\mathcal{RF}(U, L)$ is isomorphic to the lattice of closed rough sets of the approximation space $(U \times J, R)$, where $R = E \times \triangle_J$.*

I formalized and proved the following propositions regarding lattice properties.

Proposition 3.13. *The lattice L is isomorphic to a principal ideal in $\mathcal{RF}(U, L)$.*

Corollary 3.14. *$\mathcal{RF}(U, L)$ is a (completely) distributive lattice, if and only if L is a (completely) distributive lattice. In particular, $\mathcal{RF}(U, [0, 1])$ is a completely distributive lattice.*

Proposition 3.15. *(i) The lattice $\mathcal{RF}(U, L)$ is pseudocomplemented if and only if L is pseudocomplemented.*
(ii) $\mathcal{RF}(U, L)$ is an (L_n)-lattice if and only if L is an (L_n)-lattice.
(iii) If $L = (B, \vee, \wedge, \neg, 0, 1)$ is a Boolean lattice or $L = ([0, 1],\ max(,),\ min(,))$, then $\mathcal{RF}(U, L)$ is a Stone algebra where $(f, g)^*(x) = (\neg g(x), \neg g(x))$, respectively $(f, g)^*(x) = (g(x)^*, g(x)^*)$.

Theorem 3.16. *Assume that $E \subseteq U \times U$ is different from the identity relation. Then the following assertions are equivalent:*
(i) $\mathcal{RF}(U, L)$ is a three-valued Łukasiewicz algebra with $\nabla(f_1, f_2) := (f_2, f_2)$, $(f_1, f_2) \in \mathcal{RF}(U, L)$.
(ii) $\mathcal{RF}(U, L)$ is a Nelson algebra
(iii) $\mathcal{RF}(U, L)$ is a Kleene algebra
(iv) L is a Boolean lattice.

Proposition 3.17. *Let E be an equivalence relation on $U \neq \emptyset$ and L a complete atomistic Boolean lattice. Then the three-valued Łukasiewicz algebra $(\mathcal{RF}(U, L), \vee, \wedge \simeq, \nabla, (\mathbf{0}, \mathbf{0}), (\mathbf{1}, \mathbf{1})$ is isomorphic to the rough set algebra $(RS(U \times A, R), \vee, \wedge, \sim, \nabla, (\emptyset, \emptyset), (U, U))$ of the approximation space $(U \times A, R)$, where $R = E \times \triangle_A$.*

3.3.3 Novelty and Applicability

The analysis of rough L-fuzzy sets using lattice-theoretical tools yielded new results. These results prepared the investigation of the more general case when both the reference set and the approximation space are fuzzy. Furthermore, many useful properties are inherited by the lattice of rough L-fuzzy sets from the lattice L. This makes it easier for future research to examine the logical structures that can be defined on them.

3.3.4 III. Thesis

I stated and proved a representation theorem that characterizes when a pair of L-fuzzy sets corresponds to a rough L-fuzzy set over a given (crisp) approximation space. I established a connection between rough L-fuzzy sets and rough relations. I demonstrated which properties of lattice L are inherited by the lattice of rough L-fuzzy sets. I provided equivalent conditions for the lattice of rough L-fuzzy sets to form a three-valued Łukasiewicz algebra.

My publication concerning this thesis: [S10]

3.4 Fuzzy rough sets with fuzzy reference sets

3.4.1 Theoretical Overview

An *implicator* is a binary operation (mapping) $\rhd\colon [0,1]^2 \to [0,1]$ that is decreasing in the first and increasing in the second argument and that satisfies the boundary conditions

$$0 \rhd 0 = 0 \rhd 1 = 1 \rhd 1 = 1 \text{ and } 1 \rhd 0 = 0.$$

$\rhd$ is called a *border implicator* if $1 \rhd x = x$ holds for all $x \in [0,1]$.
Let $\odot$ be a t-norm and $\rhd$ be a border implicator on $[0,1]$.

Definition 3.18. *Let (U,θ) be a fuzzy approximation space. Then for any fuzzy set $f \in \mathcal{F}(U)$ its* lower approximation *$\underline{\theta}(f)$ and its* upper approximation *$\overline{\theta}(f)$ are defined as:*

$$\underline{\theta}(f)(x) := \bigwedge\{\theta(x,y) \rhd f(y) \mid y \in U\}, \text{ for all } x \in U.$$
$$\overline{\theta}(f)(x) := \bigvee\{\theta(x,y) \odot f(y) \mid y \in U\}, \text{ for all } x \in U.$$

The fuzzy rough set *of f is identified by the pair $(\underline{\theta}(f), \overline{\theta}(f)) \in \mathcal{F}(U) \times \mathcal{F}(U)$.*

Definition 3.19. *Let (U,θ) be a fuzzy approximation space, $a, b \in U$ and $F = \overline{\theta}(f)$, $G = \underline{\theta}(g)$. Then*
(i) $(a,b) \in R(F) \Leftrightarrow F(a) = \theta(a,b) \odot F(b)$;
(ii) $(a,b) \in \varrho(G) \Leftrightarrow G(a) = \theta(a,b) \rhd G(b)$.

Furthermore, consider the following equivalence relations, $E(F)$ and $\varepsilon(G)$:

$$E(F) := R(F) \cap R(F)^{-1} \text{ and } \varepsilon(G) := \varrho(G) \cap \varrho(G)^{-1}.$$

For a fuzzy approximation space (U, θ) let us introduce the notations:

$$\mathrm{Fix}(\underline{\theta}) = \{f \in \mathcal{F}(U) \mid \underline{\theta}(f) = f\}, \ \mathrm{Fix}(\overline{\theta}) = \{f \in \mathcal{F}(U) \mid \overline{\theta}(f) = f\}.$$

Additionally, I introduced the following condition:

(ID) Let $\odot$ be a left-continuous t-norm and $\rhd$ the R-implicator induced by it, or n an involutive negator, $\oplus$ a t-conorm n-dual to $\odot$ and $\rhd$ the S-implicator corresponding to them. If θ is $\odot$-transitive, then for any $f, g \in \mathcal{F}(U)$ we have $\overline{\theta}(\overline{\theta}(f)) = \overline{\theta}(f)$ and $\underline{\theta}(\underline{\theta}(g)) = \underline{\theta}(g)$ (see [17], [54], [84]). In other words, for $F = \overline{\theta}(f)$ and $G = \underline{\theta}(g)$ we have $F = \overline{\theta}(F)$ and $G = \underline{\theta}(G)$.

3.4.2 Main Results

Investigating fuzzy rough sets I stated and proved many significant propositions and theorems. One of the main results is the following representation theorem.

Theorem 3.20. *Assume that conditions in (ID) are satisfied and let (U, θ) be a fuzzy approximation space with a $\odot$-similarity relation θ of a finite range, and $F, G \in \mathcal{F}(U)$. Then (F, G) is a fuzzy rough set induced by a fuzzy set with a finite range, if and only if the following conditions hold:*

(1) $G \in Fix(\underline{\theta})$, $F \in Fix(\overline{\theta})$, $G \leq F$, and F and G have finite ranges;

(2) If $\mathcal{E}$ is a maximal $\varepsilon(G)$ class such that each $\{a\} \subseteq \mathcal{E}$ is a maximal $E(F)$ class, then there exists an element $u \in \mathcal{E}$ such that $G(u) = F(u)$;

(3) If E is a maximal $E(F)$ class such that each $\{a\} \subseteq E$ is a maximal $\varepsilon(G)$ class, then there exists an element $v \in E$ such that $G(v) = F(v)$.

In the proof of this theorem, I provided a construction for a fuzzy set f satisfying $F = \overline{\theta}(f)$ and $G = \underline{\theta}(f)$.

The lattice-theoretic examination yielded the following theorem that I stated and proved.

Theorem 3.21. *Let $\theta \colon U \times U \to L$ be a similarity relation of a finite range, and assume that $x \odot y = min(x, y)$, $x \rhd y := max(n(x), y)$, and n is an involutive negator satisfying $n(L) \subseteq L$.*

(i) If the fuzzy sets $\bigwedge_{i \in I} f_i$, $\bigwedge_{i \in I} \overline{\theta}(f_i)$, $f_i \in \mathcal{F}(U, L)$, $i \in I$ *have finite ranges, then the infimum of fuzzy rough sets* $\big(\underline{\theta}(f_i), \overline{\theta}(f_i)\big)$, $i \in I$ *exists in* $(\mathcal{FR}(U, L), \leq)$ *and its components have finite ranges.*

(ii) $(\mathcal{H}, \leq) = (\{\big(\underline{\theta}(f), \overline{\theta}(f)\big) \mid f \in \mathcal{F}_{fr}(U, L)\}, \leq)$ *is a lattice.*

(iii) If U *or* L *is finite, then* $(\mathcal{FR}(U, L), \leq)$ *is a complete lattice.*

3.4.3 Novelty and Applicability

The order structures of fuzzy rough sets with fuzzy reference sets is a less-examined part of the literature, so the representation theorem and the lattice-theoretic examination are both novel results. The analysis of this general case opens up a lot of research possibilities, e.g. combining with my previous results, I have a foundation for a future research plan on the investigation of multigranular fuzzy rough sets.

3.4.4 IV. Thesis

In the more general framework—namely, with a fuzzy approximation space and fuzzy reference sets—I stated and proved a representation theorem that specifies the conditions under which a fuzzy set pair corresponds to a fuzzy rough set, assuming a fuzzy similarity relation with a finite range. In the constructive proof I also provided the method to determine a fuzzy set whose lower and upper approximations match the given fuzzy set pair. Additionally, I determined the conditions under which fuzzy rough sets form a lattice, or even a complete lattice.

My publications concerning this thesis: [S9, S1]

3.5 Application of fuzzy rough sets for image compression

3.5.1 Theoretical Overview and the Previous Method

The image compression method proposed by Petrosino and Ferone provide a block coding procedure on gray-scale images. The fuzzy set corresponding to a pixel is based on its luminosity. Thus, the black value gets a membership value of 0, and the white value gets a membership value of 1. The method itself consists of three phases: building the codetable, encoding, decoding.

Building the codetable can be regarded as something like "training" the model. During this phase, the algorithm processes some known images. First, the image is partitioned into uniformly sized blocks (e.g. 4×4), then we move a smaller sliding window (of size e.g. 2×2) through every pixel of the block.

A few steps of this can be seen on Figure 3.1. The window represents the equivalence class of a (crisp) equivalence relation. For this class, we calculate the membership values of the lower and upper approximations and store them in a vector.

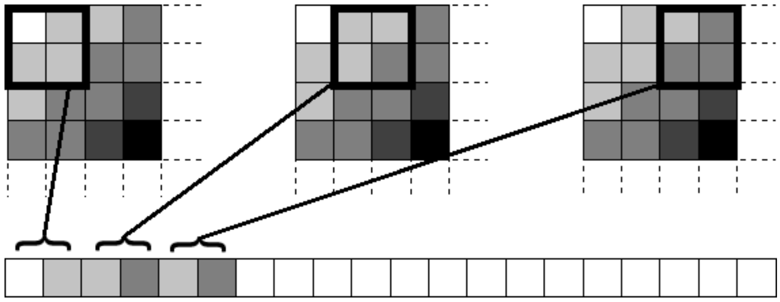

Figure 3.1: Building a vector of the codetable by sliding a window.

After processing all the training images, for each of their blocks we get a corresponding vector. This set of vectors forms the codetable. Since this can get quite large, it is recommended to perform vector quantization on this vector set.

After building the underlying model, i.e. we are done with building the codetable, we can provide an unknown image for encoding. First, similarly to the previous phase, we process the blocks of the image and calculate the corresponding vectors. Then for each vector, we find the closest one to it from the vectors in the codetable. In the compressed image, only the index of this vector will be stored.

During decoding, we need to reconstruct the original image from the list of indices in the compressed image. For this, we find the corresponding vectors in the codetable. A single pixel can be regarded as the intersection of four equivalence classes in which the corner of the sliding window is the pixel itself.

We can read the lower and upper approximations for these four classes. Using the product defined on rough fuzzy sets, we get the lower and upper approximations of the luminosity of the pixel. Assuming local smoothness, we get the decoded pixel value by averaging these.

3.5.2 Main Results

The main improvement of my proposed method comes from using fuzzy rough sets, meaning that instead of a sliding window, I use a fuzzy similarity relation. For this to work, I needed to generalize the definition of the product for rough fuzzy sets to work on fuzzy rough sets:

Definition 3.22. *Let $\theta_1, \theta_2 : U \times U \to [0,1]$ fuzzy similarity relations. Let $\sigma = \inf\{\theta_1, \theta_2\}$, furthermore, let $(\underline{f}, \overline{f}) \in \mathcal{RF}(U, \theta_1)$ and $(\underline{g}, \overline{g}) \in \mathcal{RF}(U, \theta_2)$ be fuzzy rough sets. Then the* product *of $(\underline{f}, \overline{f})$ and $(\underline{g}, \overline{g})$ is a fuzzy rough set*

$(\underline{h}, \overline{h})$ *in approximation space* (U, σ), *such that*

$$\begin{aligned}\underline{h}(x) &= \bigwedge \left\{\max(1-\sigma(x,y), \underline{f}(y), \underline{g}(y)) \mid y \in U\right\}, \\ \overline{h}(x) &= \bigvee \left\{\min(\sigma(x,y), \overline{f}(y), \overline{g}(y)) \mid y \in U\right\}.\end{aligned} \tag{3.5}$$

It is practical to choose a fuzzy similarity relation with a distane-based membership function, e.g. for pixels q and r:

$$\theta(q,r) = \max\left(0, 1 - w \cdot d(q,r)\right). \tag{3.6}$$

Instead of the 2×2 sliding window (corresponding to a crisp equivalence relation), when building the codetable or encoding an image, we use this distance-based θ fuzzy relation. During the decoding process, we get a lower and an upper approximation for each pixel. However, these are not the lower and upper approximations of the pixel yet. We need to apply the generalized product to get those. After we get them, taking the average of the lower and the upper approximation membership value, we get the color value of the reconstructed image.

I implemented both the original method and my proposed method. I compared the PSNR values of the reconstructed images according to Equation 2.6. I chose 4×4 as block size for both methods, and for my proposed method, I chose θ as the one in Equation 3.6 with $w = 0.75$, using Euclidean distance. During each run of the test I selected five images randomly. I used four of them to build the codetable, and I encoded and decoded the last one. I performed 20 runs.

The PSNR values from the test runs are shown in Table 3.1. It is apparent that in most of the cases, my proposed method provides a better PSNR value, i.e. the reconstructed image from my algorithm is more similar to the original image than the old algorithm. Only a single run (in the table, it is run number 13) resulted in a better PSNR value for the original method of Petrosino and Ferone. However, in this case, the values were really close.

Test number	PSNR (old method)	PSNR (new method)
1	28.2225	29.9399
2	25.2634	29.2819
3	24.7308	27.1482
4	23.8007	27.4343
5	21.9129	26.3070
6	24.8420	28.6957
7	20.5667	24.1792
8	26.8563	31.5673
9	30.9175	34.9920
10	27.7519	30.6260

Test number	PSNR (old method)	PSNR (new method)
11	24.9529	28.8005
12	19.3297	21.8729
13	31.4335	31.3789
14	20.3734	25.8441
15	28.4270	29.8161
16	25.1752	27.4108
17	22.2726	25.7903
18	20.5832	24.4854
19	25.2152	28.1125
20	25.5951	27.8098

Combined PSNR value for the new method:	**27.1310**
Combined PSNR value for the old method:	**23.7300**

Table 3.1: Comparison of PSNR values from the test runs.

3.5.3 Novelty and Applicability

Since the earlier image compression method was based on rough fuzzy sets, my proposed method using fuzzy rough sets and a generalized product is a novel approach. According to the experimental test runs, the proposed method consistently outperforms the old one. This gives rise to motivation for further examination: choosing better parameters, a different distance function, or a different fuzzy similar relation the improvement may be enhanced.

3.5.4 V. Thesis

I developed an image compression procedure based on fuzzy rough sets as an improvement of a previously proposed method based on rough fuzzy sets. The procedure I propose is based on the more general fuzzy rough set model, for which I also generalized the product defined over rough fuzzy sets. I implemented both the old and the new method, and I compared the quality of the compressed image in multiple test runs. According to the results, my proposed method consistently outperforms the original method, generally resulting in a compressed image with better PSNR value.

My publications concerning this thesis: [S2, S3]

4 Summary

4.1 Summary and future research directions

In my research, I studied more general variants of the rough set model primarily from a lattice-theoretic perspective. One of the generalized models arises from the concept of multigranular rough sets, where approximations are determined not by a single equivalence relation but by multiple ones. I analyzed two approaches from a lattice-theoretic viewpoint: the optimistic and the pessimistic variants. I generalized the notion of coherent relations to n relations in two different ways, and using the resulting conditions, I identified when the optimistic and pessimistic multigranular rough sets form a complete lattice, or even a completely distributive lattice. Furthermore, since multigranular rough sets, in general, are only partially ordered sets and not necessarily lattices, I also investigated the Dedekind–MacNeille completion of the order structure formed by the optimistic multigranular rough sets. I also presented several applications in the fields of information systems and recommender systems.

There exist several approaches to combining the rough set model with the fuzzy set model, depending on whether the reference set or the approximation space (the underlying relation) is treated as fuzzy. I first analyzed the case where the reference set is crisp but the approximation space is fuzzy. I examined how the core or support of fuzzy lower and upper approximations relate to the (traditional) lower and upper approximations derived from the core or support of the underlying fuzzy relation. I also showed that the lattice of fuzzy rough sets with a crisp reference set is isomorphic to the lattice of rough sets derived from the core of the fuzzy relation. Furthermore, I proved that in the case of such fuzzy rough sets, the exact fuzzy sets coincide with the exact sets of the support of the fuzzy relation, more precisely, the membership function of the former matches the characteristic function of the latter.

In the case of rough fuzzy sets, the approximation space is crisp while the reference set is fuzzy. I examined a more general case, since I considered the resulting algebraic structure not only on the $[0,1]$ interval but on a lattice L. I showed that in the case of rough L-fuzzy sets induced by an equivalence relation $E \subseteq U \times U$, the properties of lattice L are inherited by $\mathcal{RF}(U,L)$: if L is a pseudocomplemented lattice belonging to class (L_n), then $\mathcal{RF}(U,L)$ is also pseudocomplemented and belongs to the same class; if L is a Heyting algebra, then $\mathcal{RF}(U,L)$ is also a Heyting algebra; if L has the structure of a De Morgan algebra, then a De Morgan algebra can also be induced on $\mathcal{RF}(U,L)$, and vice versa. These properties essentially follow from the fact that $\mathcal{RF}(U,L)$ is isomorphic to a special subdirect product of the lattices $L^{[2]}$ and L. I also investigated under what conditions a three-valued Łukasiewicz algebra can be defined on the lattice of rough L-fuzzy sets, established connections between rough L-fuzzy sets and rough relations, and elaborated on the specific properties of $\mathcal{RF}(U,[0,1])$. I also showed that the lattice of rough L-fuzzy sets forms a complete sublattice of $\mathcal{IF}(U, L)$, i.e. the lattice of interval-valued fuzzy sets. Additionally, I formulated a representation theorem that characterizes when a pair of L-fuzzy sets (f, g) corresponds to the lower and upper approximations of a fuzzy set.

In a more general fuzzy rough set model, both the approximation space and the reference set are fuzzy. Based on the lower and upper approximations, I defined two quasi-orders. Using the equivalence relations derived from them, I provided a representation theorem for fuzzy rough sets that characterizes the conditions under which a pair of fuzzy sets corresponds to a fuzzy rough set. I also discussed the properties of the quasi-orders and the derived equivalence relations in detail. Moreover, I formulated conditions whose satisfaction implies that the fuzzy rough sets form a lattice, and I studied its properties.

In addition, I presented an image compression application, which is an improved version of a method published in 2009 [73]. The original procedure is based on the model of rough fuzzy sets, which is essentially used for feature extraction, then quantization is applied to the obtained feature vectors. The image reconstruction is carried out using the product defined on rough fuzzy sets. The method I proposed is based on the more general fuzzy rough set model, in which not only the pixel itself is considered a fuzzy set (by its brightness or color value), but the distance between pixels also defines a fuzzy similarity relation. The procedure places the fuzzy lower and upper approximations determined for each pixel into a vector for each block, then performs a reduction (quantization) on this set of vectors. During decoding, I use a more general product defined on fuzzy rough sets. The test runs demonstrate that the quality of the compressed image is consistently better using my proposed method in terms of PSNR value. In the future I would like to perform more extensive testing, where I examine the effect of changing the parameters, the distance function or the fuzzy similarity relation.

The topics discussed in the dissertation reveal several directions for further research. In the future, I would like to characterize the order structures of combined models of multigranular rough sets and fuzzy sets. Regarding the fuzzy rough set model, I intend to investigate the extent to which the finiteness assumptions used in the current results can be relaxed. If the fuzzy rough sets form a lattice, it is not necessarily distributive. Thus, it remains an open question under what conditions fuzzy rough sets are distributive. I also plan to further elaborate on the application possibilities discussed for multigranular rough sets and propose a variant combined with fuzzy methods. Additionally, I would like to explore other potential applications, including generalized (fuzzy and/or multigranular) models of decision-theoretic rough sets [111] and game-theoretic rough sets [32]. Finally, I aim to extend the presented image compression method to video compression by incorporating the time dimension.

4.2 Theses

I. Thesis

I investigated the order structures of optimistic and pessimistic multigranular rough sets. I determined under what conditions they form complete lattices and completely distributive lattices. I showed that the increasing representation of optimistic multigranular rough sets coincides with the Dedekind–MacNeille completion. I identified the conditions under which pessimistic and optimistic multigranular rough sets form a completely distributive regular double Stone lattice. (The classical Pawlakian rough sets also form such a lattice.)

My publications concerning this thesis: [S7, S6, S5]

II. Thesis

I showed that the lattice of fuzzy rough sets with crisp reference sets and defined over a fuzzy similarity relation with a dually well-ordered spectrum forms a complete lattice, which is isomorphic to the lattice of rough sets defined over the core of the fuzzy relation. I examined the relationship between the support and core of the lower and upper approximations of such fuzzy rough sets, and the approximations defined over the support and core of the fuzzy relation. I proved that the exact sets of the support of the fuzzy similarity relation—more precisely, their characteristic functions—coincide with the exact fuzzy rough sets (with a crisp reference set).

My publications concerning this thesis: [S8, S4]

III. Thesis

I stated and proved a representation theorem that characterizes when a pair of L-fuzzy sets corresponds to a rough L-fuzzy set over a given (crisp) approximation space. I established a connection between rough L-fuzzy sets and rough relations. I demonstrated which properties of lattice L are inherited by the lattice of rough L-fuzzy sets. I provided equivalent conditions for the lattice of rough L-fuzzy sets to form a three-valued Łukasiewicz algebra.

My publications concerning this thesis: [S10]

IV. Thesis

In the more general framework—namely, with a fuzzy approximation space and fuzzy reference sets—I stated and proved a representation theorem that specifies the conditions under which a fuzzy set pair corresponds to a fuzzy rough set, assuming a fuzzy similarity relation with a finite range. In the constructive proof I also provided the method to determine a fuzzy set whose lower and upper approximations match the given fuzzy set pair. Additionally, I determined the conditions under which fuzzy rough sets form a lattice, or even a complete lattice.

My publications concerning this thesis: [S9, S1]

V. Thesis

I developed an image compression procedure based on fuzzy rough sets as an improvement of a previously proposed method based on rough fuzzy sets. The procedure I propose is based on the more general fuzzy rough set model, for which I also generalized the product defined over rough fuzzy sets. I implemented both the old and the new method, and I compared the quality of the compressed image in multiple test runs. According to the results, my proposed method consistently outperforms the original method, generally resulting in a compressed image with better PSNR value.

My publications concerning this thesis: [S2, S3]

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